Convex duality and Orlicz spaces in expected utility maximization

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10th Bachelier Congress, Trinity College
Dublin, 17 July 2018

Outline

- A new look at utility maximization for arbitrary U
- New Fenchel duality result with very interesting economic interpretation
- Links to Kramkov and Schachermayer (1999, 2003)
- New approach for obtaining integral representation of optimal strategy
- Separation theorem via conjugate duality (without non-empty interiors)
- The work motivates study of Orlicz-based FTAP
- Available from `http://arxiv.org/abs/1711.09121`

Motivation I

- Self-financing strategy H
- Wealth is a stochastic integral

$$B + H \cdot S_t := B + \int_{(0,t]} H_s dS_s$$

- B is initial wealth / random endowment
- Semimartingale S represents price of d risky assets
- Stochastic integration theory requires $H \in L(S)$
- Harrison and Kreps (1979) note things go wrong when all trading strategies in $L(S)$ are allowed
- Solution: use "tame" strategies, wealth bounded from below

$$\mathscr{T}_\infty := \{H \in L(S) \mid (-H \cdot S \vee 0)^*_T \in L^\infty\}$$

Motivation II

- This works well also for utility functions with domain bounded below (e.g. log)

$$u(B) := \sup_{H \in \mathscr{T}_\infty} E[U(B + H \cdot S_T)] = \max_{H \in \mathscr{T}_\infty} E[U(B + H \cdot S_T)]$$

- Problem #1: For general U the set $\mathscr{T}_\infty$ is too restrictive -> work with wider class of tame strategies $\mathscr{T}_U$ instead
- Problem #2: Maximizer typically does not belong to $\mathscr{T}_U$
- We need "admissible strategies" $\mathscr{A}_U$ such that

$$u(B) = \sup_{H \in \mathscr{T}_U} E[U(x + H \cdot S_T)] = \sup_{H \in \mathscr{A}_U} E[U(B + H \cdot S_T)]$$
$$= \max_{H \in \mathscr{A}_U} E[U(B + H \cdot S_T)]$$

Two streams

- Literature has two streams
  - Utility functions finite on a half-line (solved)
  - Utility functions finite on the whole $\mathbb{R}$ (some gaps left)
  - Not clear how the two fit together
- Random endowment B
- When B is constant we write $x = B$

Utility finite on half-line

- Solved by Kramkov and Schachermayer (1999, 2003)
- S need not be locally bounded
- Assume no arbitrage, take $\mathscr{A} = \mathscr{T}_\infty$
- KS99: Optimal strategy in all probabilistic models for all $x \in \mathbb{R}$ iff U has reasonable asymptotic elasticity, RAE($+\infty$)

$$\lim_{x \to +\infty} \frac{xU'(x)}{U(x)} < 1$$

- KS03: Optimal strategy in a given probab. model for all $x \in \mathbb{R}$ iff indirect utility satisfies

$$\lim_{x \to +\infty} \frac{u(x)}{x} = 0$$

Utility functions finite on the whole $\mathbb{R}$

- Problem #1: Optimizer not in $\mathscr{T}_\infty$, cf. Black-Scholes model
- Solution: Schachermayer (2001) X is admissible terminal wealth if there are $H^n \in \mathscr{T}_\infty$ such that

$$U(x + H^n \cdot S_T) \stackrel{L^1(P)}{\to} U(X)$$

- Now S must be locally bounded
- U must satisfy RAE($-\infty$) to guarantee optimizer in all models for all x

$$\lim_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1$$

U finite on $\mathbb{R}$: admissible trading strategies

- Problem #2: we have optimal terminal wealth but where is the optimal strategy H?

$$X = x + H \cdot S_T$$

- Delbaen et al. (2002), Kabanov and Stricker (2002)

$$H \in \mathscr{A} \iff H \cdot S \text{ is a martingale under all local martingale measures}$$

- Works well for exponential utility with S locally bounded
- Schachermayer (2003) shows in general we need

$$H \in \mathscr{A} \iff H \cdot S \text{ is a supermartingale under all local martingale measures}$$

U finite on $\mathbb{R}$: beyond locally bounded S

- Problem #3: Utility maximization makes sense for S with unbounded jumps
- Biagini and Frittelli (2005, 2007, 2008) define tame strategies via bigger Orlicz space $L^{\hat{U}}$

$$\mathscr{T}_{U,W} := \{H \in L(S) \mid (-H \cdot S \vee 0)^*_T \leq W \in L^{\hat{U}}\}$$

  - $\hat{U}$ is left tail of U; it always holds that

$$L^\infty \hookrightarrow L^{\hat{U}} \hookrightarrow L^1$$

  - $\mathscr{A}$ is the supermartingale class (dual definition)
  - S is not locally bounded but must be compatible with $\hat{U}$
- Solution may in principle depend on the loss control W

Inspiration

- Quadratic utility: Č. and Kallsen (2007)
  - $U(x) = x - x^2/2$
  - S is (locally) in L^2
- Kallsen's definition of admissibility – purely under P (no duality)
- $\mathscr{T}$: buy-and-hold (locally), bounded volume
- H is admissible iff there is sequence of tame H^n such that
  - $H^n \cdot S_t \to H \cdot S_t$ in probability for all t
  - $H^n \cdot S_T \to H \cdot S_T$ in $L^2(P)$

Inspiration II

- Biagini and Č. (2011) general U
- Admissibility: swap L^2-convergence for

$$U(x + H^n \cdot S_T) \overset{L^1(P)}{\to} U(x + H \cdot S_T)$$

- Walter got there first. But we have approximations at intermediate times.
- Tame strategies controlled from both sides

$$\mathscr{T} := \{H \in L(S) \mid (H \cdot S)^*_T \in L^{\hat{U}}\}$$

- Compatibility requirement $S \in \mathscr{S}^{\hat{U}}_\sigma$ i.e. $|S|^*_T \in L^{\hat{U}}$ σ-locally
- Dual optimizer is a σ-martingale measure

$$\mathcal{M} := \{Q \ll P \mid S \text{ is a } Q\text{-}\sigma\text{-martingale}\}$$

Fast forward to 2018

- Assume $B \in L^{\hat{U}}$ throughout
- Tame strategies with max loss in $L^{\hat{U}}$ (control from below)

$$\mathscr{T} := \{H \in L(S) \mid (-H \cdot S \vee 0)^*_T \in L^{\hat{U}}\}$$

- No compatibility requirement on S
- Admissible strategies: modify intermediate convergence

$$(\operatorname*{rqlim}_{n\to\infty} H^{(n)} \cdot S)_t := \lim_{q \searrow t, q \in \mathbb{Q}} (\lim_{n\to\infty} H^{(n)} \cdot S_q)$$

- Require $H \cdot S = \operatorname{rqlim}_{n\to\infty} H^{(n)} \cdot S$ only on the set

$$U(B + H \cdot S_T) < U(\infty)$$

Conjugate duality

- Follow Rockafellar (1974)
- For concave f define concave conjugate function f^*

$$f^*(y) = \inf_x \{\langle x, y \rangle - f(x)\}$$

- For convex f define convex conjugate

$$f^*(y) = \sup_x \{\langle x, y \rangle - f(x)\}$$

- Meaning of $\langle x, y \rangle$ depends on the context:
  - $\langle x, y \rangle = xy$ when $x, y \in \mathbb{R}$
  - $\langle X, Y \rangle = E[XY]$ when X, Y are random variables
  - $\langle x, y \rangle = y(x)$ when y is understood as a linear functional
- Effective domain is also overloaded: for concave f

$$\operatorname{dom} f := \{x : f(x) > -\infty\}$$

Economic duality

- Take U and compute its conjugate U^*
- Let $V := -U^*$
- Let $\hat{V}$ be the right tail of V; we also have $\hat{V} = \hat{U}^*$
- The space of all Y such that $\langle X, Y \rangle = E[XY]$ is finite for all $X \in L^{\hat{U}}$ is precisely $L^{\hat{V}}$ (Zaanen, 1983)
- To represent prices as expectations one must work in duality $(L^{\hat{U}}, L^{\hat{V}})$
- RAE$(-\infty)$ on U is equivalent to $\Delta_2(+\infty)$ on V and $\hat{V}$
- Therefore RAE$(-\infty)$ on U implies $L^{\hat{V}} = M^{\hat{V}}$
- $$\begin{aligned} M^{\hat{V}} &= \text{Orlicz heart (Edgar and Sucheston, 1989)} \\ &= \text{Morse-Transue space (Morse and Transue, 1950)} \end{aligned}$$

Orlicz spaces as modular spaces

- Integral functional I_φ

$$I_\varphi(X) := E[\varphi(X)]$$

- $I_{\hat{U}}$ plays the role of so-called modular
  - $X \in L^{\hat{U}} \iff I_{\hat{U}}(\lambda X)$ for some $\lambda > 0$;
  - $X \in M^{\hat{U}} \iff I_{\hat{U}}(\lambda X)$ for all $\lambda > 0$
- modular convergence $I_{\hat{U}}(\lambda(X_n - X)) \to 0$ for some $\lambda > 0$;
- gauge norm $\|X\|_{\hat{U}} = \inf\{\lambda > 0 \mid I_{\hat{U}}(X/\lambda) \leq 1\}$;
- More general modular space: Musielak-Orlicz
  - accomodates state-dependent utility
- Further generalization: Banach lattice theory
  - Original space L (here $L^{\hat{U}}$); dual space $L_n^{\sim}$ (here $L^{\hat{V}}$)
  - Modular on $L_n^{\sim}$ defined by conjugation (here $I_{\hat{V}}$)

Topology and duality I

- Nowak (1989)
  - Strong topology $\beta(L^{\hat{U}}, L^{\hat{V}})$ coincides with norm topology
  - Modular topology coincides with Mackey topology $\tau(L^{\hat{U}}, L^{\hat{V}})$
- $\beta(L^{\hat{U}}, L^{\hat{V}})$ may not be compatible with economic duality
- The following are equivalent
  - modular $I_{\hat{U}}$ is metrizing; i.e. $I_{\hat{U}}(X_n) \to 0 \implies I_{\hat{U}}(2X_n) \to 0$;
  - $X_n \to 0$ modularly $\implies X_n \to 0$ in norm;
  - modular and strong topology coincide;
  - $L^{\hat{U}}$ is barrelled in modular topology;
  - $L^{\hat{V}}$ is norm-dual of $L^{\hat{U}}$; $(L^{\hat{U}})^\star = L^{\hat{V}}$
- If, additionally the probability space is not finite
  - $M^{\hat{U}} = L^{\hat{U}}$
- If, additionally the probability space is not purely atomic
  - $\hat{U}$ satisfies $\Delta_2(+\infty)$; i.e. there is $K > 0$ and $x_0 > 0$ such that

$$\hat{U}(2x) \leq K\hat{U}(x) \text{ for all } x > x_0$$

Topology and duality II

- We need three conjugation symbols
- Norm (strong) dual $(L^{\hat{U}})^{\star}$; conjugate $f^{\star}$
- Economic (modular) dual $L^{\hat{V}}$; conjugate $f^{\circledast}$
  - compatible topologies between $\sigma(L^{\hat{U}}, L^{\hat{V}})$ and $\tau(L^{\hat{U}}, L^{\hat{V}})$
- In KS99,03 $L^{\hat{U}} = (L^{\hat{V}})^{\star}$
- In general this is not true but one has $L^{\hat{U}} = (M^{\hat{V}})^{\star}$
- Weak$*$ dual $M^{\hat{V}}$ (for $L^{\hat{U}} \not\approx L^1$); conjugate f^*
  - $*$-u.s.c. characterized by sequences Gao and Xanthos (2018)
- $(L^{\hat{U}})^{\star} \hookleftarrow L^{\hat{V}} \hookleftarrow M^{\hat{V}}$ ($\star$ strong, $\circledast$ modular, $*$ weak-star)
- Connections to criteria for relative $\sigma(L^{\varphi}, L^{\varphi^*})$-compactness (Andô, 1962; Nowak, 1993)

How easy is it to have $M^{\hat{V}} \subsetneq L^{\hat{V}}$?

- All commonly used utility functions have $\hat{V} \in \Delta_2(+\infty)$
- If $\hat{V}$ grows faster than power then for sure $\hat{V} \notin \Delta_2(+\infty)$
- Converse is not true: for any $p \geq 1$ and $\varepsilon > 0$ there is $\hat{V}$ such that Salekhov (1968)
  - $L^{p+\varepsilon} \hookleftarrow L^{\hat{V}} \hookleftarrow L^p$
  - $\hat{V}$ does not satisfy Δ_2

Market completion, separating measure

- $\mathcal{K}$ cone of tame terminal wealths with zero initial capital,

$$\mathcal{K} := \{H \cdot S_T : H \in \mathscr{T}\},$$

- $\mathcal{C}$ convex cone of super-replicable claims,

$$\mathcal{C} := \left(\mathcal{K} - L^0_+\right) \cap L^{\hat{U}}.$$

- Each $Q \in P_{\hat{V}}$ can be understood as a complete market with

$$\mathcal{C}_Q := \{X \in L^{\hat{U}} \mid E^Q[X] \equiv \langle X, dQ/dP \rangle \leq 0\}.$$

- We say Q is a full completion / separating measure if $\mathcal{C} \subseteq \mathcal{C}_Q$
- Geometrically dQ/dP is in the economic polar of $\mathcal{C}$

$$\mathcal{C}^{\circledast} := \{Y \in L^{\hat{V}} \mid \langle X, Y \rangle \equiv E[XY] \leq 0 \text{ for all } X \in \mathcal{C}\}.$$

Complete market utility

- For $L^{\hat{U}} \sim L^{\infty}$ assume $\operatorname{ess\,inf} B$ is high enough
- Every bliss-free complete market satisfies

$$u_Q(B) := \sup_{X \in \mathcal{C}_Q} I_U(B+X) = \min_{\lambda>0} \left\{ I_V(\lambda dQ/dP) + \lambda E^Q[B] \right\}$$

- Max utility with budget constraint expressed via Q Pliska (1986)
- The set of bliss-free complete markets

$$P_V = \{Q \mid \lambda dQ/dP \in \operatorname{dom} I_V \text{ for some } \lambda > 0\}$$

- Bliss-free complete markets for U bounded above

$$P_{\hat{V}} = \{Q \mid dQ/dP \in L_+^{\hat{V}}\}$$

Duality over separating measures

- Since $\mathcal{C}_Q \supseteq \mathcal{C}$ we have $u_Q(B) \geq u(B)$
- We look for completions that increase utility only a little

$$u(B) = \inf_{dQ/dP \in \mathcal{C}^{\circledast}} u_Q(B)$$

- Important for construction of optimal trading strategy (later)
- Rewrite as

$$u = \sup_{X \in \mathcal{C}} I_U(B+X) = \inf_{Y \in \mathcal{C}^{\circledast}} \{I_V(Y) + E[YB]\}$$

- Rewrite again to obtain Fenchel duality, $\mathcal{A} := B + \mathcal{C}$

$$\sup_{X \in L^{\hat{U}}} \{I_U(X) - \delta_{\mathcal{A}}(X)\} = \inf_{Y \in L^{\hat{V}}} -\left\{I_U^{\circledast}(Y) - \delta_{\mathcal{A}}^{\circledast}(Y)\right\},$$

- Always holds if $\circledast = \star$

First new result, $\mathcal{A} = B + \mathcal{C}$

- Fenchel duality is subtle, it is only easily available in $\star$ form

$$u(B) = \sup_{X \in L^{\hat{U}}} \{I_U(X) - \delta_{\mathcal{A}}(X)\} = \min_{Y \in (L^{\hat{U}})^\star} -\{I_U^\star(Y) - \delta_{\mathcal{A}}^\star(Y)\}$$

- Optimal Y may contain singular parts
- $\mathcal{D} := \operatorname{dom} I_U$, the set of wealth distributions with finite utility
- Assume $\mathcal{A} \cap \operatorname{core} \mathcal{D} \neq \emptyset$, automatic for $B \in M^{\hat{U}}$
- We prove, assuming nothing else!!!,

$$u(B) = \sup_{X \in L^{\hat{U}}} \{I_U(X) - \delta_{\mathcal{A}\cap\mathcal{D}}(X)\} = \min_{Y \in L^{\hat{V}}} -\left\{I_U^\circledast(Y) - \delta_{\mathcal{A}\cap\mathcal{D}}^\circledast(Y)\right\}$$

- Now $\hat{Y}$ is a random variable (no singular parts)
- If $\mathcal{A} \cap \mathcal{D}$ is a strict subset of $\mathcal{A}$ then $\hat{Y}$ need not be in $\mathcal{C}^\circledast$
- So $\hat{Y}$ may not correspond to a separating measure

The link to FTAP, $B = 0$

- I_U is known to be $\circledast$-u.s.c.; (even $*$-u.s.c. when $L^{\hat{U}} \not\sim L^\infty$)
- Intuitively utility should not increase from $\mathcal{C}$ to $\mathcal{C}^{\circledast\circledast}$
- For sure it does not increase from $\mathcal{C}$ to $(\mathcal{C} \cap \mathcal{D})^{\circledast\circledast}$
- Likewise it does not increase from $\mathcal{C}$ to $\mathcal{C}^{\star\star}$
  - In the nice case $\mathcal{C}^{\star\star}$ equals $\mathcal{C}^{\circledast\circledast}$
- But, increase from $\mathcal{C}$ to $\mathcal{C}^{\circledast\circledast}$ remains possible if $\star \neq \circledast$
- If this happens, goodbye duality over pricing measures
- $L^{\hat{U}}$-FTAP to rescue (small market): $\mathcal{C} = \mathcal{C}^{\circledast\circledast}$
  - Excluding $L^{\hat{U}} \sim L^\infty$, this is an open question,
  - Cuchiero and Teichmann (2015) revisit L^∞-FTAP
  - In large market $\mathcal{C} = \mathcal{C}^{\circledast\circledast}$ may not hold
  - It is disconnected from absence of arbitrage
- The $\mathcal{C} = \mathcal{C}^{\circledast\circledast}$ in classical L^∞-FTAP is a happy coincidence

Utility increases from $\mathcal{C}$ to $\mathcal{C}^{\circledast\circledast}$

- We now understand how this happens
- It is an example of core $A \cap \operatorname{dom} f \neq \emptyset$, f convex l.s.c., where f decreases on the closure of A;
  - this cannot happen on a barrelled space
  - so never on a metric or normed space and never in finite dim
- Construction (generic on not purely atomic space):
  - this is a "large market" construction
  - take X_n that converges to zero modularly but not in norm;

$$I_{\hat{U}}(X_n) \to 0 \text{ but } I_{\hat{U}}((1+\varepsilon)X_n) = \infty$$

  - give X_n zero mean;
  - take arbitrage-free Z disjointly supported from X_n;
  - make Z satisfy $I_U(2Z) > I_U(Z)$;
  - this implies $I_U(Z + X_n) \to I_U(Z)$ but $I_U(2Z + 2X_n) = -\infty$

Modular Fenchel duality continues to hold

- Modular Fenchel duality holds separately for $\mathcal{C}$ and $\mathcal{C}^{\circledast\circledast}$

$$
\begin{aligned}
u = \sup_{X\in L^{\hat{U}}} \{I_U(X) - \delta_{\mathcal{C}\cap\mathcal{D}}(X)\} &= \min_{Y\in L^{\hat{V}}} -\{I_U^{\circledast}(Y) - \delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(Y)\} \\
\leq \sup_{X\in L^{\hat{U}}} \{I_U(X) - \delta_{\mathcal{C}^{\circledast\circledast}\cap\mathcal{D}}(X)\} &= \min_{Y\in L^{\hat{V}}} -\{I_U^{\circledast}(Y) - \delta^{\circledast}_{\mathcal{C}^{\circledast\circledast}\cap\mathcal{D}}(Y)\}
\end{aligned}
$$

Market completion (type 2 = effective)

- $(B+\mathcal{C})\cap\mathcal{D}\subseteq x+B+\mathcal{C}_Q$ for some $x\geq 0$, smallest such x
- Boils down to separating measure when $x=0$
- When $x>0$ we have only an effective completion
- EXAMPLE 1: $\max_{H\in\mathbb{R}} E[\ln\left(1+H(e^R-1)\right)]$; $R\sim N(\mu,1)$
  - Cook it up so that $\hat{H}=1$ is optimal
  - FOC: $E[e^{-R}(e^R-1)]>0$ for μ sufficiently high
  - Dual optimizer $\hat{Y}=e^{-R}$; $\delta^{\circledast}_{\mathcal{C}\cap\mathcal{D}}(\hat{Y})=E[e^{-R}(e^R-1)]>0$

Implication for supermartingale deflators

- Separating measures are naturally supermartingale measures
- Also, their density process is a supermartingale deflator
- Effective completions are not supermartingale measures
- Their density process is not a supermartingale deflator
- In KS99,03 effective completion can be turned into terminal value of a supermartingale deflator
- EXAMPLE 2: Exponential utility; Lévy process
  - positive jumps like tempered stable
  - negative jump Poisson with jump size $-1/2$
  - sufficiently negative drift to make optimal investment short, and, "up to the hilt"
  - again corner solution - constant dollar amount short
  - no supermart. deflator with terminal value $\hat{Y}$ possible

New construction for optimal strategy

- Forget the dual optimizer (think truncated utility)
  - monotone mean-variance preferences Maccheroni et al. (2009)
  - link to truncated quadratic utility Č. et al (2012)
- Construct candidate strategy H under one $Q \in \mathcal{M} \cap P^e_{\hat{V}}$
- Supermartingale compactness result Delbaen and Schachermayer (1998, Theorem D)
- Utility of $B + H \cdot S_T$ is at least $u(B)$
- Prove $H \cdot S$ is a supermartingale for every $Q \in \mathcal{M} \cap P_{\hat{V}}$
- Squeeze down the utility of H using duality over $\mathcal{C}^{\circledast}$
  - need densities in $\mathcal{M} \cap P_{\hat{V}}$ to be $\| \cdot \|_{\hat{V}}$-dense in $\mathcal{C}^{\circledast}$
- The difficult part is to prove no duality gap over $\mathcal{C}^{\circledast}$

Norm-coercivity in losses

Definition

Expected utility I_U is norm-coercive in losses on a set $\mathcal{G} \subseteq L^{\hat{U}}$ if

$$\lim_{\|X^-\|_{\hat{U}} \to \infty, X \in \mathcal{G}} I_U(X) = -\infty.$$

Lemma

Consider $\mathcal{G} \subseteq L^{\hat{U}}$ and suppose there is $\tilde{Y} \in L^{\hat{V}}$ such that

1. $\lambda\tilde{Y} \in \operatorname{dom} I_V$ *for two distinct values of* $\lambda > 0$;
2. $\{E[X\tilde{Y}]\}_{X \in \mathcal{G}}$ *is bounded from above.*

Then I_U is norm-coercive in losses on $\mathcal{G}$.

Uniform integrability for utility of upside wealth

- Define indirect utility of norm-bounded endowment

$$\overline{u}(x) := \sup_{\|B\|_{\hat{U}} \leq x} u(B) = \sup\{I_U(X+Z) \mid X \in \mathcal{C}, \|Z\|_{\hat{U}} \leq x\}$$

- Note $\overline{u}(x) = u(x)$ for $L^{\hat{U}} \sim L^{\infty}$
- Subset of terminal wealth distributions

$$\mathcal{Z}(k_1, k_2) := \{X + Z \mid X \in \mathcal{C}, Z \in L^{\hat{U}}, \|X^-\|_{\hat{U}} \leq k_1, \|Z\|_{\hat{U}} \leq k_2\}$$

Lemma

Condition $\lim_{x \to \infty} \overline{u}(x)/x = 0$ *implies that the set* $\{U(\mathcal{Z}(k_1, k_2)^+)\}$ *is uniformly integrable for every* $k_1, k_2 > 0$.

Optimal terminal wealth

Proposition (Optimal Wealth)

Assume

1. $(B+\mathcal{C}) \cap \operatorname{core} \mathcal{D} \neq \emptyset$;
2. *no arbitrage over* $\mathcal{C}^{\circledast\circledast}$;
3. $\lim_{x\to\infty} \overline{u}(x)/x = 0$;
4. *dual optimizer* $\hat{Y}$ *satisfies* $\lambda\hat{Y} \in \operatorname{dom} I_V$ *for some* $\lambda > 1$ *(automatic when* $M^{\hat{V}} = L^{\hat{V}}$*).*

Then there is sequence $\{X_n\} \in \mathcal{C}$ *with* $I_U(B+X_n) \nearrow u(B) < U(\infty)$ *and r. v.* $\hat{X}$ *(not necessarily in* $L^{\hat{U}}$*) such that* $X_n \overset{P\text{-a.s.}}{\to} \hat{X}$,

$$U(B+\hat{X}) - (B+\hat{X})\hat{Y} = V(\hat{Y}),$$

$$U(B+X_n) \overset{L^1(P)}{\to} U(B+\hat{X}).$$

Duality over separating measures

Theorem (Duality Over Separating Measures)

Assume either i) $M^{\hat{U}} = L^{\hat{U}}$*; or ii)*

- $\mathcal{C} = \mathcal{C}^{**}$*; no arbitrage over* $\mathcal{C}$*;*
- $\lim_{x\to\infty} \overline{u}(x)/x = 0$*;*
- *and, only in the case* $L^{\hat{U}} \not\approx L^{\infty}$*, there is* $\tilde{Y} \in \mathcal{C}^{\circledast}$ *such that* $\lambda\tilde{Y} \in \operatorname{dom} I_V$ *for two distinct values of* $\lambda \geq 0$*.*

Then duality over separating measures holds for all $B \in L^{\hat{U}}$*.*

Proof.

By Rockafellar (1974) enough to show $u^{**}(B) = u(B)$. Will prove more, $u^{**} = u$, using Gao and Xanthos (2018). □

Weaker (automatic) result: $\sup_{X\in\mathcal{C}^{**}} I_U(X) = (I_V \Box \delta_{-\mathcal{C}^*})^{**}(0)$.

Optimal strategy

Proposition (Candidate Optimal Strategy)

Assume there is $\bar{Q} \in \mathcal{M} \cap P^e_{\hat{V}}$. Under the assumptions of Optimal Wealth Proposition there is a trading strategy $H \in L(S)$, a sequence of maximizing tame strategies $H^{(n)}$ and a semimartingale $\tilde{V}$ such that

1. $\tilde{V}$ *is* $\bar{Q}$*-supermartingale;*
2. $\tilde{V} = \operatorname{rqlim}_{n\to\infty} H^{(n)} \cdot S$;
3. $H \cdot S \geq \tilde{V}$ *and* $H \cdot S - \tilde{V}$ *is an increasing process;*
4. *In particular,* $H^{(n)} \cdot S_T \overset{P\text{-a.s.}}{\rightarrow} \tilde{V}_T$;
5. $U(B + H^{(n)} \cdot S_T) \overset{L^1(P)}{\rightarrow} U(B + \tilde{V}_T)$ *and thus* $I_U(B + \tilde{V}_T) = u(B) \in \mathbb{R}$;
6. $H \cdot S$ *is a* Q*-supermartingale for any* $Q \in \mathcal{M} \cap P_{\hat{V}}$.

Existence of optimal strategy

Theorem

*Assume (i) $B \in L^{\hat{U}}$; (ii) no arbitrage over $\mathcal{C}^{\circledast\circledast}$;
(iii) $\mathcal{C}^{\circledast}_{\sigma}$ is norm-dense in $\mathcal{C}^{\circledast}$ (automatic when $L^{\hat{U}} \sim L^{\infty}$);
(iv) $(B + \mathcal{C}) \cap \operatorname{core} \mathcal{D} \neq \emptyset$ (automatic when $L^{\hat{U}} = M^{\hat{U}}$); and
(v) further assumptions as listed below. Then the Candidate Optimal Strategy H is optimal and admissible.*

	$M^{\hat{V}} = L^{\hat{V}}$	$M^{\hat{V}} \subsetneq L^{\hat{V}}$
$M^{\hat{U}} = L^{\hat{U}}$	none required;	$\lambda\hat{Y} \in \operatorname{dom} I_V$ for $\lambda > 1$;
$M^{\hat{U}} \subsetneq L^{\hat{U}} \nsim L^{\infty}$	$\mathcal{C} = \mathcal{C}^{\circledast\circledast}$;	$\mathcal{C} = \mathcal{C}^{**}$; $\lambda\hat{Y} \in \operatorname{dom} I_V$ for $\lambda > 1$;
$M^{\hat{U}} \subsetneq L^{\hat{U}} \sim L^{\infty}$	none required	cannot occur

Literature I

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