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Simplified stochastic calculus via semimartingale representations

In memory of Peter Carr

Aleš Černý

Bayes Business School

11th Bachelier Congress, Hong Kong
16 June 2022

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Calculus of predictable variations with applications in finance

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- Simplified calculus from several perspectives
  - As a recipe for obtaining new formulae
  - As a useful shorthand
  - As a tool for practical calculations
  - As an algorithmic device
  - As a pedagogical tool
- Will touch upon

Carr, P. and D. B. Madan (1999). Option valuation using the fast Fourier transform. *J. Comput. Finance 2*(4), 61–73.

Carr, P. and R. Lee (2013). Variation and share-weighted variation swaps on time-changed Lévy processes. *Finance Stoch. 17*(4), 685–716.

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Based on a series of joint works with Johannes Ruf (LSE Math.)

- Intro for readers familiar with $\mathrm{d}t, \mathrm{d}W, \mathrm{d}N$ calculus
  - Simplified stochastic calculus with applications in Economics and Finance, *European J. Oper. Res. 293*(2), 2021, 547–560, ssrn:3500384.
  - Appendix on affine Riccati equations à la Duffie, Pan, and Singleton, ssrn:3752072.
- Theory at the level of Jacod and Shiryaev
  - Pure-jump semimartingales. *Bernoulli 27*(4), 2021, 2624–2648, arXiv:1909.03020.
  - Simplified stochastic calculus via semimartingale representations. *Electron. J. Probab. 27*, 2022, paper no. 3, ssrn:3633638.
  - Multiplicative compensators and changes of measure, ssrn:3633622.
- Slides available from www.martingales.sk

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- Semimartingale X ($\mathbb{R}$-valued for now)
- Function $\xi : \mathbb{R} \to \mathbb{R}$ in $\mathcal{C}^2$ with $\xi(0) = 0$
- Want to formalize a process with increment $\xi(\mathrm{d}X)$
- Suppose X has jumps of finite variation
- Continuous part $\mathrm{d}X^c := \mathrm{d}X - \Delta X$

$$\xi(\mathrm{d}X_t) = \xi'(0)\mathrm{d}X_t^c + \frac{1}{2}\xi''(0)\mathrm{d}[X^c, X^c]_t + \xi(\Delta X_t)$$

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- Now make the formula universal

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$$\xi(\mathrm{d}X_t) = \xi'(0)\mathrm{d}X_t + \frac{1}{2}\xi''(0)\mathrm{d}[X^c, X^c]_t + \xi(\Delta X_t) - \xi'(0)\Delta X_t$$

- Now make the formula universal
  - add jumps to the first term and subtract them in the last term

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- Now make the formula universal
  - add jumps to the first term and subtract them in the last term
  - reinterpret continuous quadratic covariation

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- Now make the formula universal
  - add jumps to the first term and subtract them in the last term
  - reinterpret continuous quadratic covariation
- Denote the resulting process started at 0 by $\xi \circ X$

Émery formula II

- "G-variation" in Carr and Lee (2013)!

Definition 2.2 (G-variation) For $G \in \mathbb{V}(Y)$, define the G-variation of Y to be

$$V_t^{Y,G} := \alpha_G \mathrm{TV}(Y^{\mathrm{d}})_t + \beta_G (Y_t - Y_0) + \gamma_G [Y^{\mathrm{c}}]_t + \sum_{0<s\le t} \big(G(\Delta Y_s) - \beta_G \Delta Y_s\big), \quad (2.8)$$

($G \circ Y$; $G'(0)$; $1/2\, G''(0)$; $G'(0)$)

Proposition 2.3 *For any $G \in \mathbb{V}(Y)$ and any sequence $\Delta_n \to 0$, let*

$$V^{Y,G}(n)_t := \sum_{j=1}^{\lfloor t/\Delta_n \rfloor} G(Y_{j\Delta_n} - Y_{(j-1)\Delta_n}).$$

For any *decomposition of the form (2.2), the $V^{Y,G}$ defined by (2.8) satisfies*

$$V^{Y,G}(n) \longrightarrow V^{Y,G}$$

in probability in the Skorokhod sense, as $n \to \infty$. Jacod (2008)

- Émery (1978) showed UCP convergence, same formula

$$\xi \circ X = D\xi(0) \cdot X + \frac{1}{2} D^2\xi(0) \cdot [X,X]^c + (\xi - D\xi(0)\,\mathrm{id}) * \mu^X$$

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- Now make ξ / G predictable
- Hence calculus of predictable variations
- Integral is a "linear variation"

$$\zeta \cdot X = (\zeta \,\mathrm{id}) \circ X$$

- Why are general variations a relatively unused concept?
  1. For continuous processes, all variations are linear–quadratic
  2. Expression $\xi \circ X$ arises often but in contexts that have nothing to do with variations, e.g., Goll and Kallsen (2000, Lemma A.8)

$$\mathcal{L}(S) := \int_0^{\cdot} \frac{\mathrm{d}S_u}{S_u} = (\mathrm{e}^{\mathrm{id}} - 1) \circ \ln S,$$

     or Doléans-Dade (1970) $\mathscr{E}(X) = \mathrm{e}^{\ln(1+\mathrm{id})\circ X}$.
  3. $\xi \circ X$ is a variation only if ξ jointly left-continuous, work in progress, related to functional Itô formula and Itô-Wentzell formula, see Č. and Ruf (2021a)

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- Want the calculus to be rigorous, flexible, and easy to use
- Need a rich class of ξ / G where nothing strange can happen

Definition (Universal representing functions)

$\mathfrak{U}$ denotes the set of predictable functions ξ such that, P–a.s.,

(i) $\xi_t(0) = 0$, for all $t \geq 0$.

(ii) $x \mapsto \xi_t(x)$ is twice real-differentiable at zero, for all $t \geq 0$.

(iii) $D\xi(0)$ and $D^2\xi(0)$ are locally bounded.

(iv) There is a predictable locally bounded process $K > 0$ such that

$$\sup_{0<|x|\leq 1/K} \frac{\left|\xi(x) - D\xi(0)x\right|}{|x|^2} \text{ is locally bounded.}$$ □

Key properties

- $\mathfrak{U}$ is closed under common operations
- Starting from $X = X_0 + \mathrm{id} \circ X$ and using only
  - composition, i.e., the "$\circ$" operation with functions in $\mathfrak{U}$;
  - change of variables by means of (deterministic) $\mathcal{C}^2$ functions;
  - locally bounded integration;

  every result will be of the form $\xi \circ X$ for some $\xi \in \mathfrak{U}$
- In practice, we never leave $\mathfrak{U}$, so no checking necessary

Key properties II

- Jump transformation

$$\Delta(\xi \circ X) = \xi(\Delta X)$$

- Initial value makes no difference to the "$\circ$" operation

$$\xi \circ X = \xi \circ (X - X_0)$$

- $\mathrm{id}, \mathrm{id}^2 \in \mathfrak{U}$ with

$$\mathrm{id} \circ X = X - X_0; \qquad (\mathrm{id}\,\mathrm{id}^\top) \circ X = [X, X]$$

- If X is continuous, then

$$\xi \circ X = \left(\xi'(0)\,\mathrm{id} + \frac{1}{2}\xi''(0)\,\mathrm{id}^2\right) \circ X$$

Complex values and restricted domains

- Definitions of $\mathfrak{U}$ and "$\circ$" handle restricted domains
  - $\xi \circ X$ is a semimartingale if $\xi \in \mathfrak{U}$ and $\xi(\Delta X)$ is finite
  - for example, $\log(1 + \mathrm{id}) \in \mathfrak{U}$
- $\mathfrak{U}$ and "$\circ$" handle $\mathbb{C}^n$–valued ξ applied to $\mathbb{C}^d$–valued X
  - Define lifts $\hat{\mathrm{id}} : \mathbb{C}^m \to \mathbb{R}^{2m}$ and $\check{\mathrm{id}} : \mathbb{C}^m \to \mathbb{C}^{2m}$

$$\hat{\mathrm{id}} = (\mathrm{Re}\,\mathrm{id}_1, \mathrm{Im}\,\mathrm{id}_1, \ldots, \mathrm{Re}\,\mathrm{id}_m, \mathrm{Im}\,\mathrm{id}_m)$$
$$\check{\mathrm{id}} = (\mathrm{id}_1, \mathrm{id}_1^*, \ldots, \mathrm{id}_m, \mathrm{id}_m^*)$$

  - Real derivatives $\hat{D}$; Wirtinger derivatives $\check{D}$

$$\xi \circ X = \hat{D}\xi(0) \cdot \hat{X} + \frac{1}{2}\hat{D}^2\xi(0) \cdot [\hat{X}, \hat{X}]^c + (\xi - \hat{D}\xi(0)\hat{\mathrm{id}}) * \mu^X$$
$$= \check{D}\xi(0) \cdot \check{X} + \frac{1}{2}\check{D}^2\xi(0) \cdot [\check{X}, \check{X}]^c + (\xi - \check{D}\xi(0)\check{\mathrm{id}}) * \mu^X$$

Beyond universal representations

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- One can move beyond universal representations
- $\mathfrak{I}(X)$ are functions specific to X such that $\xi \circ X$ makes sense
- E.g., $\zeta \in L(X)$ not locally bounded: $\zeta\,\mathrm{id} \notin \mathfrak{U}$ but $\zeta\,\mathrm{id} \in \mathfrak{I}(X)$
- See main theory paper
- Main takeaway: It is ok to have lots of universal functions on the outside and one specific function on the inside

Simplified calculus: an example

- Suppose X is log return
- Cumulative rate of return is then $\mathcal{L}(e^X)$
- Let us compute $[\mathcal{L}(e^X), \mathcal{L}(e^X)]$
- Proceed in steps
  - Change of variables
$$e^X = e^{X_0} + (e^{X_- + \mathrm{id}} - e^{X_-}) \circ X$$
  - Locally bounded integration
$$\mathcal{L}(e^X) = e^{-X_-} \cdot e^X = e^{-X_-}(e^{X_- + \mathrm{id}} - e^{X_-}) \circ X = (e^{\mathrm{id}} - 1) \circ X$$
  - Composition
$$[\mathcal{L}(e^X), \mathcal{L}(e^X)] = \mathrm{id}^2 \circ \mathcal{L}(e^X) = (e^{\mathrm{id}} - 1)^2 \circ X$$
- Instead of manipulating complicated **stochastic** expressions ✘ one performs simple **algebraic** operations ✓

Alternatives are unappetizing

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- Consider classical notation for a Lévy process

$$X_t = X_0 + \int_0^t \alpha \mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s + \int_0^t \int_{|x|\leq 1} x\widehat{N}(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{|x|>1} xN(\mathrm{d}s, \mathrm{d}x),$$

  - $\widehat{N}(\mathrm{d}t, \mathrm{d}x) = N(\mathrm{d}t, \mathrm{d}x) - \Pi(\mathrm{d}x)\mathrm{d}t$
  - N is a Poisson jump measure
  - Π the corresponding Lévy measure
  - $\alpha, \sigma \in \mathbb{R}$
- Imagine the previous slide in this notation

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- Power variations

$$\mathrm{id}^3 \circ X = [[X,X],X]$$
$$\mathrm{id}^4 \circ X = [[X,X],[X,X]] = [[[X,X],X],X] \text{ etc.}$$

- Stochastic log and natural exponential

$$\mathcal{L}\left(\mathrm{e}^X\right) = (\mathrm{e}^{\mathrm{id}} - 1) \circ X$$

- Principal value log and stochastic exponential

$$\mathscr{E}(X) = \mathrm{e}^{\log(1+\mathrm{id})\circ X}, \qquad \Delta X \neq -1 \text{ Doleáns-Dade formula}$$
$$\log|\mathscr{E}(X)| = \log|1+\mathrm{id}| \circ X, \qquad \Delta X \neq -1$$
$$\log \mathscr{E}(X) = \log(1+\mathrm{id}) \circ X, \qquad \mathrm{Re}\,\mathscr{E}(X) > 0$$

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- Generalized Yor formulae

$$\mathscr{E}(X)^\alpha = \mathscr{E}\left(((1+\mathrm{id})^\alpha - 1) \circ X\right),$$

$$|\mathscr{E}(X)|^\alpha = \mathscr{E}\left((|1+\mathrm{id}\,|^\alpha - 1) \circ X\right),$$

- In two variables

$$\mathscr{E}\big(X^{(1)}\big)^\alpha \mathscr{E}\big(X^{(2)}\big)^\beta = \mathscr{E}\left(((1+\mathrm{id}_1)^\alpha (1+\mathrm{id}_2)^\beta - 1) \circ X\right)$$
$$\big|\mathscr{E}\big(X^{(1)}\big)\big|^\alpha \big|\mathscr{E}\big(X^{(2)}\big)\big|^\beta = \mathscr{E}\left(\left(|1+\mathrm{id}_1|^\alpha |1+\mathrm{id}_2|^\beta - 1\right) \circ X\right)$$

- Special cases in literature: $\alpha = \beta = 1$ (Yor, 1976); $\alpha = 1, \beta = -1$ (Mémin, 1978); $\alpha = 0, \beta = -1$ (Larsson and Ruf, 2019)

Useful identities II

- Generalized Yor formulae

$$\mathscr{E}(X)^\alpha = \mathscr{E}\left(((1+\mathrm{id})^\alpha - 1)\circ X\right), \qquad \begin{array}{l}\alpha \in \mathbb{Z}_{++}; \text{ or}\\ \alpha \in \mathbb{Z}_-;\ \Delta X \neq -1; \text{ or}\\ \alpha \in \mathbb{C}\setminus\mathbb{Z};\ \operatorname{Re}\mathscr{E}(X) > 0\end{array}$$

$$|\mathscr{E}(X)|^\alpha = \mathscr{E}\left((|1+\mathrm{id}\,|^\alpha - 1)\circ X\right),$$

- In two variables

$$\mathscr{E}\big(X^{(1)}\big)^\alpha \mathscr{E}\big(X^{(2)}\big)^\beta = \mathscr{E}\left(\left((1+\mathrm{id}_1)^\alpha(1+\mathrm{id}_2)^\beta - 1\right)\circ X\right)$$
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- Generalized Yor formulae

$$\mathscr{E}(X)^\alpha = \mathscr{E}\left(((1+\mathrm{id})^\alpha - 1)\circ X\right),$$

$$|\mathscr{E}(X)|^\alpha = \mathscr{E}\left((|1+\mathrm{id}\,|^\alpha - 1)\circ X\right), \qquad \begin{array}{l}\alpha \in (0,\infty); \text{ or} \\ \alpha \in \mathbb{C}\setminus\mathbb{R}_+;\ \Delta X \neq -1\end{array}$$

- In two variables

$$\mathscr{E}\big(X^{(1)}\big)^\alpha \mathscr{E}\big(X^{(2)}\big)^\beta = \mathscr{E}\left(((1+\mathrm{id}_1)^\alpha (1+\mathrm{id}_2)^\beta - 1)\circ X\right)$$
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- Converse of generalized Yor formulae

$$\mathcal{L}(X^\alpha) = ((1+\mathrm{id})^\alpha - 1) \circ \mathcal{L}(X),$$

$$\mathcal{L}(|X|^\alpha) = (|1+\mathrm{id}\,|^\alpha - 1) \circ \mathcal{L}(X),$$

- In two variables

$$\mathcal{L}\left(X^\alpha Y^\beta\right) = \left((1+\mathrm{id}_1)^\alpha (1+\mathrm{id}_2)^\beta - 1\right) \circ \left(\mathcal{L}(X), \mathcal{L}(Y)\right)$$

$$\mathcal{L}\left(|X|^\alpha |Y|^\beta\right) = \left(|1+\mathrm{id}_1|^\alpha |1+\mathrm{id}_2|^\beta - 1\right) \circ \left(\mathcal{L}(X), \mathcal{L}(Y)\right)$$

- Special case

$$\mathcal{L}\left(\frac{X}{Y}\right) = \left(\frac{1+\mathrm{id}_1}{1+\mathrm{id}_2} - 1\right) \circ \left(\mathcal{L}(X), \mathcal{L}(Y)\right)$$

Useful identities III

- Converse of generalized Yor formulae

$$\mathcal{L}(X^\alpha) = ((1+\mathrm{id})^\alpha - 1) \circ \mathcal{L}(X), \quad \begin{array}{l} \alpha \in \mathbb{Z};\ X, X_- \neq 0 \text{ or} \\ \alpha \in \mathbb{C} \setminus \mathbb{Z};\ \operatorname{Re} X, \operatorname{Re} X_- > 0 \end{array}$$

$$\mathcal{L}(|X|^\alpha) = (|1+\mathrm{id}\,|^\alpha - 1) \circ \mathcal{L}(X),$$

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$$\mathcal{L}(|X|^\alpha) = (|1+\mathrm{id}\,|^\alpha - 1) \circ \mathcal{L}(X), \quad \alpha \in \mathbb{C};\ X, X_- \neq 0$$

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- Standard Itô calculus is based on signal–noise decomposition
  - signal = drift (predictable FV part)
  - noise = Brownian motion (loc. martingale part)
- Two reasons why this works
  - Knowledge of drift has many applications: PDEs, expected values, HJB equation, Feynman–Kac
  - Stochastic integral of BM is not absolutely convergent
- Idea is difficult to extend cleanly to jump processes
  - Not all jump processes have a drift (only small jumps do)
  - Must split jumps into small / large; compensate small
  - Notation becomes super heavy
  - Nonetheless this is industry standard; textbooks Øksendal and Sulem (2007), Applebaum (2009)
  - So complicated that many serious papers stick to compound Poisson jumps; Duffie, Pan and Singleton (2000), Feng and Linetsky (2008), Cai and Kou (2012), Aït-Sahalia and Matthys (2019), Shiraya and Takahashi (2019)

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- We are looking at this the wrong way round
- Stochastic integrals ARE measure-invariant
  - McKean (1969) gives measure-invariant Itô formula
  - in full generality known since mid 70-ies; Meyer (1976)
- Keeping track of the drift is notationally expensive
- Paradigm change:
  - Write processes in a measure-invariant way
  - Only compute the drift when it is really needed
- Need an easy, clean way to compute the drift of $\xi \circ X$
- Émery formula already provides this!

$$\xi \circ X = \xi'(0) \cdot X^c \quad + \frac{1}{2}\xi''(0) \cdot [X, X]^c + \xi * \mu^X$$

Drift calculation II

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- Add and subtract only small jumps indicated by h

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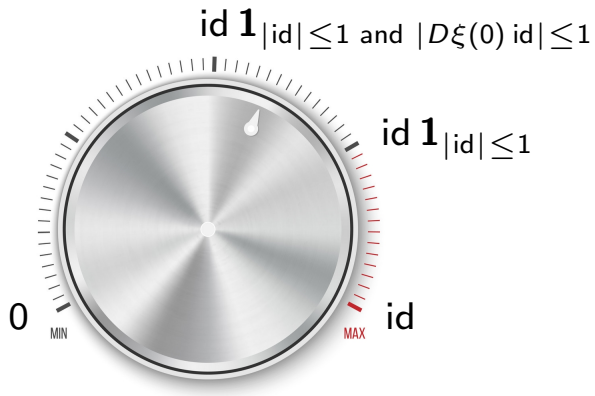
$$\xi \circ X = \xi'(0) \cdot X[h] + \frac{1}{2}\xi''(0) \cdot [X, X]^c + (\xi - \xi'(0)h) * \mu^X$$

- Add and subtract only small jumps indicated by h

Dial h for truncation

- Émery formula represents a spectrum of equivalent expressions



- There is flexibility in the choice of h
  - $h = 0$ when X has compound Poisson jumps; $X[0] = X^c$
  - $h = \mathrm{id}$ when X has finite drift; $X[\mathrm{id}] = X$
  - Otherwise $h = \mathrm{id}\,\mathbf{1}_{|\mathrm{id}|\leq 1} \in \mathfrak{U}$
- For $\xi \notin \mathfrak{U}$, $D\xi(0)$ may not be locally bounded. Then take

$$h = \mathrm{id}\,\mathbf{1}_{|\mathrm{id}|\leq 1 \text{ and } |D\xi(0)\,\mathrm{id}\,|\leq 1} \in \mathfrak{I}(X)$$

Drift calculation III

- We have $X[h] = X_0 + h \circ X$, except when $h = 0$, but always
$$X[h] = X - (\mathrm{id} - h) * \mu^X$$
- Assume $\xi \circ X$ is special. Choose h appropriately. Then,
$$\xi \circ X = \xi'(0) \cdot X[h] + \frac{1}{2}\xi''(0) \cdot [X, X]^c + (\xi - \xi'(0)h) * \mu^X$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$B^{\xi \circ X} = B^{\xi'(0) \cdot X[h]} + \frac{1}{2}\xi''(0) \cdot [X, X]^c + (\xi - \xi'(0)h) * \nu^X$$
- Additive compensator: $\xi \circ X - B^{\xi \circ X} \in \mathscr{M}_{\mathrm{loc}}$
- If $\xi \circ X$ is PII, then for all $t \geq 0$
$$\mathrm{E}[\xi \circ X_t] = B_t^{\xi \circ X}$$

Drift calculation IV

- Č. and Ruf (2020) Multiplicative compensator

$$\frac{\mathscr{E}(\xi \circ X)}{\mathscr{E}(B^{\xi \circ X})} \in \mathscr{M}_{\mathrm{loc}}$$

  $\mathbb{R}$–valued case in Lépingle and Mémin (1978)

- Č. and Ruf (2020) If $\xi \circ X$ is PII, then for all $t \geq 0$

$$\mathrm{E}[\mathscr{E}(\xi \circ X)_t] = \mathscr{E}(B^{\xi \circ X})_t.$$

  See also Cont and Tankov (2004); Kallsen and Muhle-Karbe (2010)

- Take $\xi^{(u)} = \mathrm{e}^{iu\,\mathrm{id}} - 1$, $u \in \mathbb{R}$; X Lévy process
  - $\mathscr{E}\left(B^{\xi^{(u)} \circ X}\right)_t$ is now the Lévy–Khintchin formula
  - Roll out the Fourier transform Carr and Madan (1999)
  - Not just pricing but also quadratic hedging and residuals; Hubalek, Kallsen and Krawczyk (2006), Č (2007)
  - Extend to affine processes via Riccati equations

Lévy process setting

- Classical notation

$$X_t = X_0 + \int_0^t \alpha \mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s + \int_0^t \int_{|x|\leq 1} x\widehat{N}(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{|x|>1} xN(\mathrm{d}s, \mathrm{d}x),$$

  - $\widehat{N}(\mathrm{d}t, \mathrm{d}x) = N(\mathrm{d}t, \mathrm{d}x) - \Pi(\mathrm{d}x)\mathrm{d}t$
  - N is a Poisson jump measure
  - Π the corresponding Lévy measure
  - $\alpha, \sigma \in \mathbb{R}$
- Simplified calculus: just record the triplet

$$\left(b^{X[1]} = \alpha, c^X = \sigma^2, F^X = \Pi\right).$$

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$$\left(b^{X[1]} = \alpha, c^X = \sigma^2, F^X = \Pi\right).$$

Characteristic function

- Take $v \in \mathbb{C}$
- We want to evaluate $\mathrm{E}[\mathrm{e}^{vX_t}]$
- Easily obtain universal representation

$$\mathcal{L}(\mathrm{e}^{vX}) = (\mathrm{e}^{vx} - 1) \circ X$$

- This yields

$$b^{\mathcal{L}(\mathrm{e}^{vX})} = \alpha v + \frac{1}{2}\sigma^2 v^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{vx} - 1 - vx\mathbf{1}_{|x|\leq 1}\right) \Pi(\mathrm{d}x),$$

- Lévy–Khintchin formula
- $b^{\mathcal{L}(\mathrm{e}^{vX})} t$ is cumulant generating function of $X_t - X_0$

$$\mathrm{E}[\mathrm{e}^{vX_t}] = \mathrm{e}^{vX_0} \exp\left(b^{\mathcal{L}(\mathrm{e}^{vX})} t\right)$$

Maximization of exponential utility

- $X = \ln S$
- Cumulative yield on \$1 investment $R = \mathcal{L}(e^X)$
- Fixed dollar investment λ
- Utility of terminal wealth $-\mathrm{E}[e^{-\lambda R_t}]$
- Need to find the drift of $\mathcal{L}(e^{-\lambda R})$
- Use composition rule to find

$$\mathcal{L}(e^{-\lambda R}) = \left(e^{-\lambda(e^x-1)} - 1\right) \circ X$$

- Evaluate the drift

$$b^{\mathcal{L}(e^{-\lambda R})} = -\alpha\lambda + \frac{\sigma^2}{2}(\lambda^2 - \lambda) + \int_{\mathbb{R}} \left(e^{-\lambda(e^x-1)} - 1 + \lambda x \mathbf{1}_{|x|\leq 1}\right) \Pi(dx).$$

- Expected utility is $-\exp\left(b^{\mathcal{L}(e^{-\lambda R})} t\right)$

Minimal entropy martingale measure

- Denote optimal investment by λ_*
- Density of MEMM dQ/dP is proportional to marginal utility $\mathrm{e}^{-\lambda_* R}$
- We want Q–drift of $\mathcal{L}(\mathrm{e}^{vX})$ to get c.f. of X_t under Q
- By Girsanov the same as P–drift of

$$\mathcal{L}(\mathrm{e}^{vX}) + \left[\mathcal{L}(\mathrm{e}^{vX}), \mathcal{L}(\mathrm{e}^{-\lambda_* R})\right] = \xi \circ X$$

  with $\xi = (\mathrm{e}^{vx} - 1)\mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)}$
- Evaluate the P–drift of $\xi \circ X$

$$b_{\mathrm{Q}}^{\mathcal{L}(\mathrm{e}^{vX})} = b^{\xi \circ X} = \alpha v + \frac{\sigma^2}{2}\left(v^2 - 2\lambda_* v\right) + \int_{\mathbb{R}} \left((\mathrm{e}^{vx} - 1)\mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)} - vx\mathbf{1}_{|x|\leq 1}\right) \Pi(\mathrm{d}x),$$

Change of measure in classical notation

- Is utterly horrendous
- Find an explicit expression for $\log Z$

$$\log Z = -\int_0^\cdot \lambda_* \sigma \mathrm{d}W_s - \frac{1}{2}\int_0^\cdot \lambda_*^2 \sigma^2 \mathrm{d}s + \int_0^\cdot \int_{\mathbb{R}} -\lambda_*(\mathrm{e}^x - 1)\widehat{N}(\mathrm{d}s, \mathrm{d}x)$$
$$+ \int_0^\cdot \int_{\mathbb{R}} \left(-\lambda_*(\mathrm{e}^x - 1) - \left(\mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)} - 1\right)\right) \Pi(\mathrm{d}x)\mathrm{d}s$$

- Construct a new Brownian motion for the measure Q,

$$\mathrm{d}W_t^{\mathrm{Q}} = \mathrm{d}W_t + \lambda_* \sigma \mathrm{d}t$$

- New compensated Poisson jump measure

$$\widehat{N}^{\mathrm{Q}}(\mathrm{d}t, \mathrm{d}x) = \widehat{N}(\mathrm{d}t, \mathrm{d}x) + \left(1 - \mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)}\right) \Pi(\mathrm{d}x)\mathrm{d}t$$

  - Applebaum (2009), Theorem 5.2.12, Exercise 5.2.14
  - Øksendal and Sulem (2007), Theorem 1.32 and Lemma 1.33
- Substitute for W, N, and $\hat{N}$ in the original expression for X
- More examples in this vein Č. and Ruf (2021c)

Girsanov-type results in simplified calculus

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Theorem (THM)

Assume $\mathscr{E}(Y) > 0$ *is special and that* $\mathscr{E}(Y)/\mathscr{E}(B^Y)$ *is a UI martingale. Let*

$$\frac{\mathrm{dQ}}{\mathrm{dP}} = \frac{\mathscr{E}(Y)_\infty}{\mathscr{E}(B^Y)_\infty}.$$

For Q *and a semimartingale* X*, the following are equivalent.*

(i) X *is* Q*–special.*

(ii) $X\mathscr{E}(Y)/\mathscr{E}(B^Y)$ *is special.*

(iii) $X + [X, Y]$ *is special.*

If either of these conditions holds, then $B^{X,\mathrm{Q}} = \frac{1}{1+\Delta B^Y} \cdot B^{X+[X,Y]}$ *and*

$$\mathscr{E}(B^{X,\mathrm{Q}}) = \frac{\mathscr{E}\left(B^{X+Y+[X,Y]}\right)}{\mathscr{E}(B^Y)}.$$

Multiplicative compensator under measure change

- Illustration from Lépingle and Mémin (1978)
- Take Y as above with $B^Y = 0$
- Assume $(1+\mathrm{id})\ln(1+\mathrm{id}) \circ Y$ is special
- THM yields that $\ln(1+\mathrm{id}) \circ Y$ is Q–special with

$$B^{\ln(1+\mathrm{id})\circ Y, \mathrm{Q}} = B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y}.$$

- Next, fix some $\lambda \in (0,1)$
- Observe that $\mathscr{E}^\lambda(Y) = \mathscr{E}(((1+\mathrm{id})^\lambda - 1) \circ Y)$ is special
- Hence $\mathscr{E}^{\lambda-1}(Y)$ is Q–special by THM
- By simplified calculus the following is also Q-special

$$\mathrm{e}^{(\lambda-1)(\ln(1+\mathrm{id})\circ Y - B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y})} = \mathscr{E}^{\lambda-1}(Y)\mathrm{e}^{(1-\lambda)B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y}}$$

- Observe LHS$= \mathrm{e}^M$ where $M \in \mathscr{M}_{\mathrm{loc}}(\mathrm{Q})$

Digression: the role of convexity / concavity

- LEMMA: If $M \in \mathscr{M}_{\text{loc}}$ and e^M special, then its multiplicative compensator is non-decreasing
- Proof: e^M special iff $\mathcal{L}(\mathrm{e}^M) = (\mathrm{e}^{\mathrm{id}} - 1) \circ M$ special
- Since $M \in \mathscr{M}_{\text{loc}}$, processes $(\mathrm{e}^{\mathrm{id}} - 1) \circ M$ and $(\mathrm{e}^{\mathrm{id}} - 1 - \mathrm{id}) \circ M$ have the same drift
- But $\mathfrak{U} \ni \mathrm{e}^{\mathrm{id}} - 1 - \mathrm{id} \geq 0$
- Hence $(\mathrm{e}^{\mathrm{id}} - 1 - \mathrm{id}) \circ M$ is a non-decreasing process!
- More generally $\xi \in \mathfrak{U}$ convex / concave, then

$$(\xi - \xi'(0)\,\mathrm{id}) \circ X$$

  is non-decreasing / non-increasing for any X
- $B^{\mathcal{L}(\mathrm{e}^M)}$ non-decreasing yields multiplicative compensator $\mathscr{E}(B^{\mathcal{L}(\mathrm{e}^M)})$ non-decreasing Q.E.D.

Mult. compensator under measure change II

- We conclude the Q–multiplicative compensator of

$$\mathrm{e}^{(\lambda-1)(\ln(1+\mathrm{id})\circ Y - B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y})} = \mathscr{E}^{\lambda-1}(Y)\mathrm{e}^{(1-\lambda)B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y}}$$

  is non-decreasing
- By THM, the same as (P–)multiplicative compensator of

$$\mathscr{E}^{\lambda}(Y)\mathrm{e}^{(1-\lambda)B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y}}$$

- Hence $\mathscr{E}(B^{((1+\mathrm{id})^{\lambda}-1)\circ Y})\mathrm{e}^{(1-\lambda)B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y}}$ is non-decreasing
- The non-decreasing property yields

$$\mathscr{E}(B^{((1+\mathrm{id})^{\lambda}-1)\circ Y}) \geq \mathrm{e}^{(\lambda-1)B^{(1+\mathrm{id})\ln(1+\mathrm{id})\circ Y}},$$

  cf. Lépingle and Mémin (1978), Lemma III.4, Equation (3.4).

Applications of multiplicative compensators I

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- Computation of characteristic functions
  - Jacod and Shiryaev (2003), Section III.7
- Esscher-type measure changes Kallsen and Shiryaev (2002)
  - minimax martingale measures, e.g., Goll and Rüschendorf (2001); Jeanblanc, Klöppel and Miyahara (2007)
  - martingale measures associated with ad-hoc numeraire changes, e.g., Eberlein, Papapantoleon and Shiryaev (2009)
  - non-martingale measures, e.g., the opportunity-neutral measure in MV hedging Č. and Kallsen (2007), Section 3.4
- Proofs of moment bounds
  - to show existence and uniqueness of BSDE solutions, e.g., Kazi-Tani, Possamaï and Zhou (2015), Lemma A.5
  - to estimate variation distance of probability measures Kabanov, Liptser and Shiryaev (1986), Theorem 2.1
  - to prove uniform integrability of local martingales, e.g., Lépingle and Mémin (1978), Lemma I.4; Ruf (2013), Corollary 5

Applications of multiplicative compensators II

- Filtration extension / shrinkage
  - e.g., Nikeghbali and Yor (2006), Section 4; Kardaras (2015); Aksamit and Jeanblanc (2017), Chapter 5; Kardaras and Ruf (2020), Section 5
- Theory of Markov processes
  - e.g., Itô and Watanabe (1965) Chapter 2; Chen et al. (2004), Theorem 3.1

Illustration: take $\lambda > 0$, $M \in \mathscr{M}_{\mathrm{loc}}$, $\Delta M \neq -1$, and assume $|\mathscr{E}(M)|^{-\lambda}$ is special. If $\mathscr{E}(B^{(|1+\mathrm{id}\,|^{-\lambda}-1)\circ M}) \leq C$, then $|\mathscr{E}(M)|^{-\lambda}$ is integrable.

$$|\mathscr{E}(M)|^{-\lambda} = \frac{|\mathscr{E}(M)|^{-\lambda}}{\mathscr{E}\big(B^{(|1+\mathrm{id}\,|^{-\lambda}-1)\circ M}\big)}\mathscr{E}\big(B^{(|1+\mathrm{id}\,|^{-\lambda}-1)\circ M}\big) \leq C\frac{|\mathscr{E}(M)|^{-\lambda}}{\mathscr{E}\big(B^{(|1+\mathrm{id}\,|^{-\lambda}-1)\circ M}\big)}$$

RHS is a positive local mart. and supermart. Q.E.D.

Mellin transform of signed stochastic exponential

- Cannot be tackled by existing tools
- For fixed $\alpha \in \mathbb{C}$ define

$$f_1 = |\mathrm{id}|^\alpha \mathbf{1}_{\mathrm{id}\neq 0}; \quad f_2 = |\mathrm{id}|^\alpha \left(\mathbf{1}_{\mathrm{id}>0} - \mathbf{1}_{\mathrm{id}<0}\right); \quad \xi_{1,2} = f_{1,2}(1+\mathrm{id})-1$$

- Simplified calculus yields, for all $\mathbb{R}$–valued Y,

$$f_{1,2}(\mathscr{E}(Y)) = \mathscr{E}(\xi_{1,2} \circ Y)$$

- Observe $f_1 + f_2 = (\mathrm{id}^+)^\alpha$ and $f_1 - f_2 = (\mathrm{id}^-)^\alpha$
- If Y is PII, we can get Mellin transform of $\mathscr{E}(Y)_t^+$ and $\mathscr{E}(Y)_t^-$

$$\mathrm{E}[f_{1,2}(\mathscr{E}(Y)_t)] = \mathscr{E}\left(B^{\xi_{1,2}\circ Y}\right)_t$$

- Lévy–Khintchin is of no use here
- Perform this calculation for exponential Lévy MV portfolio

MV wealth as a signed stochastic exponential I

- Merton model log return X with triplet

$$(b^{X[0]} = \mu, \sigma^2, \Pi = \lambda\Phi(0, \gamma^2))$$

- Parameter values $\mu = 0.2$, $\sigma = 0.2$, $\lambda = 1$, $\gamma = 0.1$, and zero interest rate
- Optimal wealth $1 - \mathscr{E}(\underbrace{-a(e^{\mathrm{id}} - 1) \circ X}_{Y})$, where

$$a = \frac{b^{(e^{\mathrm{id}}-1)\circ X}}{b^{(e^{\mathrm{id}}-1)^2\circ X}} = \frac{\mu + \sigma^2/2 + \lambda(e^{\gamma^2/2} - 1)}{\sigma^2 + \lambda(e^{2\gamma^2} - 2e^{\gamma^2/2} + 1)} \approx 4.48;$$

- Evaluate the exponential compensators

$$b^{\xi_1(\mathrm{id};\alpha)\circ Y} = b^{\xi_1(-a(e^{\mathrm{id}}-1);\alpha)\circ X} = I_1(\alpha);$$
$$b^{\xi_2(\mathrm{id};\alpha)\circ Y} = b^{\xi_2(-a(e^{\mathrm{id}}-1);\alpha)\circ X} = I_1(\alpha) - 2I_2(\alpha),$$

MV wealth as a signed stochastic exponential II

- Auxiliary expressions

$$I_1(\alpha) = -\alpha a\left(\mu + \frac{1}{2}(1+a)\sigma^2\right) + \frac{1}{2}\alpha^2(a\sigma)^2 + \int_{\mathbb{R}}(|1-a(e^x-1)|^\alpha \mathbf{1}_{a(e^x-1)\neq 1} - 1)\Pi(dx);$$

$$I_2(\alpha) = \int_{\mathbb{R}} |1-a(e^x-1)|^\alpha \mathbf{1}_{a(e^x-1)>1}\Pi(dx)$$

- Evaluate the Mellin transforms

$$g_+(\alpha) = \mathsf{E}\left[|\mathscr{E}(Y)_t|^\alpha \mathbf{1}_{\{\mathscr{E}(Y)_t>0\}}\right] = e^{I_1(\alpha)T}\frac{1+e^{-2I_2(\alpha)T}}{2};$$

$$g_-(\alpha) = \mathsf{E}\left[|\mathscr{E}(Y)_t|^\alpha \mathbf{1}_{\{\mathscr{E}(Y)_t<0\}}\right] = e^{I_1(\alpha)T}\frac{1-e^{-2I_2(\alpha)T}}{2}$$

- Observe $g_-(0) = \mathsf{P}(\mathscr{E}(Y) < 0) \approx 2.2\%$

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- Compute subdensities of $\log|\mathscr{E}(Y)|$ conditional on $\mathscr{E}(Y) \gtrless 0$ by Fourier inversion of conditional c.f.-s

$$\phi_+(u) = \frac{g_+(iu)}{g_+(0)}; \qquad \phi_-(u) = \frac{g_-(iu)}{g_-(0)}, \qquad u \in \mathbb{R},$$

- The whole computation is structured and algorithmic

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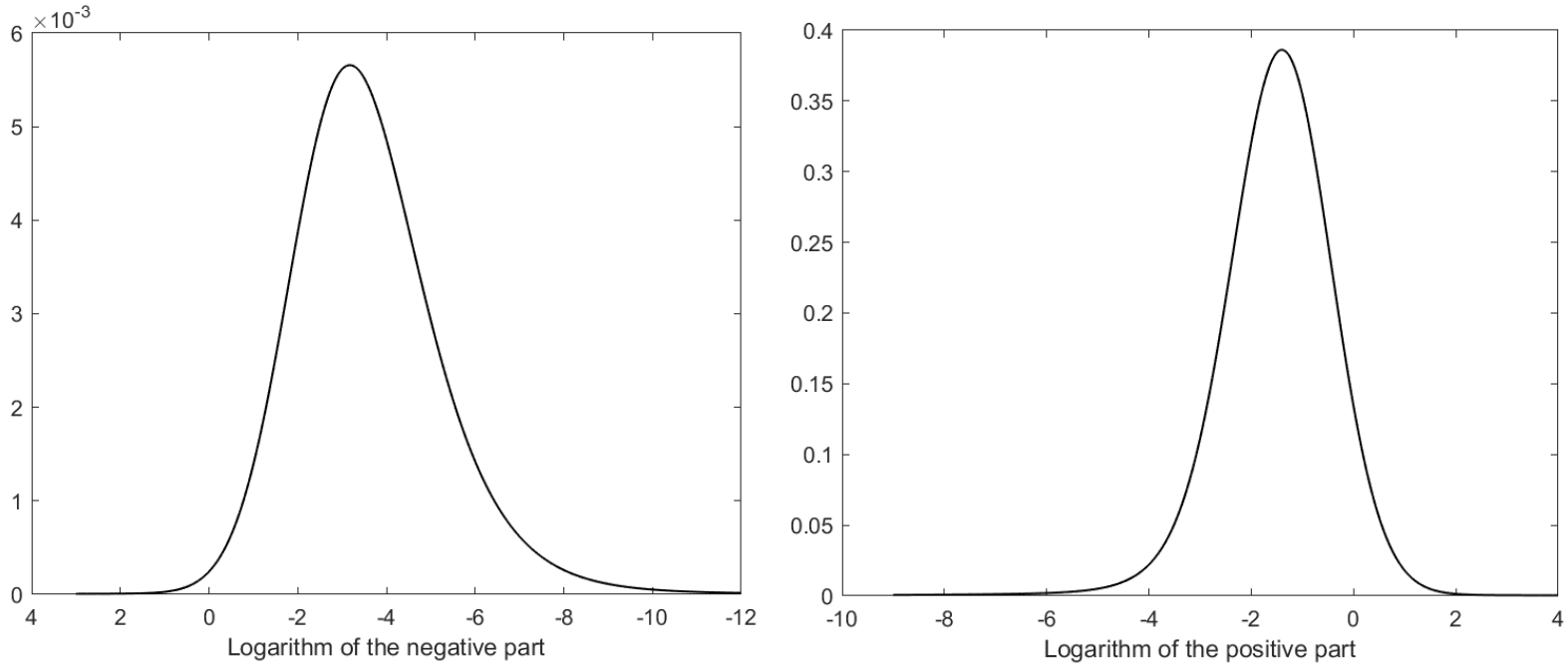

(a) Subdensity of $\log \mathscr{E}(-a(e^{\mathrm{id}} - 1) \circ X)_T^-$. (b) Subdensity of $\log \mathscr{E}(-a(e^{\mathrm{id}} - 1) \circ X)_T^+$.

Figure: Distribution of a signed stochastic exponential

MV wealth as a signed stochastic exponential V

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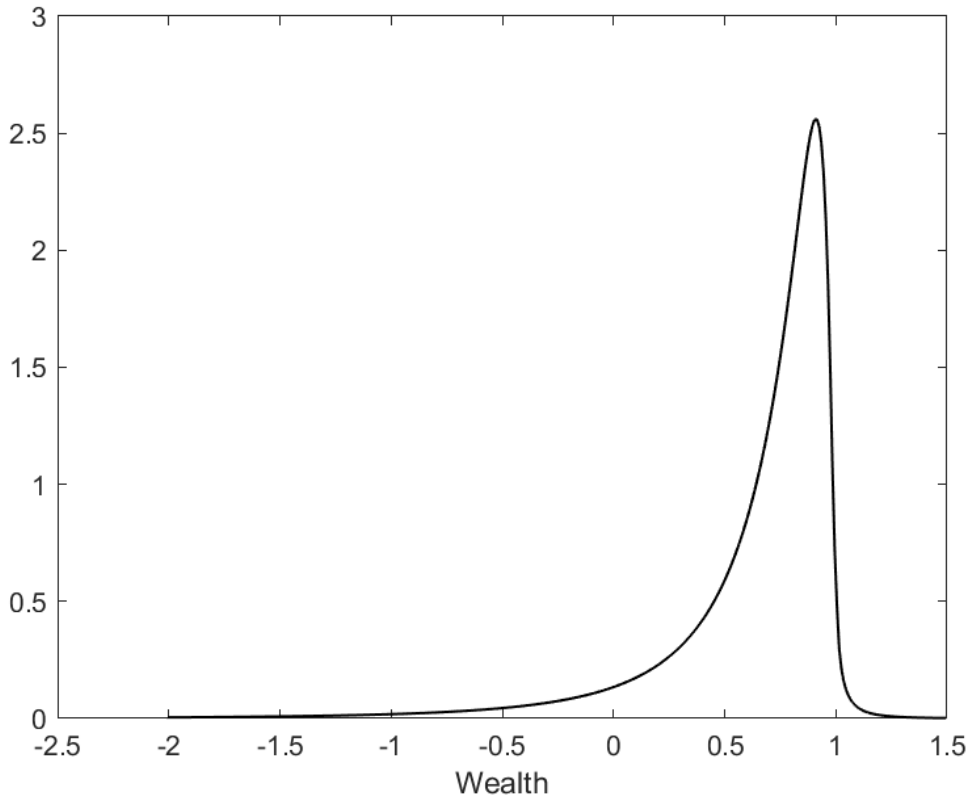

Figure: Density of the terminal wealth distribution $1 - \mathscr{E}(-a(\mathrm{e}^{\mathrm{id}} - 1) \circ X)_T$.

Pedagogical opportunities, continuous X

- For continuous X it is common to write

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2$$

- In Émery notation literally $(\mathrm{d}X_t)^2 = \mathrm{d}[X,X]_t$
- If happy to write $(\mathrm{d}X_t)^2$, then, e.g., $\ln(1+\mathrm{d}X_t)$ not so bad
- McKean (1969) suggested the heuristics $\mathrm{d}W_t\mathrm{d}t = 0$, $(\mathrm{d}t)^2 = 0$
- Better rule: $(\mathrm{d}X_t)^3 = 0$, $(\mathrm{d}X_t)^4 = 0$ for any continuous X
- Why useful: for continuous X

$$\xi \circ X = \left(\xi'(0)\,\mathrm{id} + \frac{1}{2}\xi''(0)\,\mathrm{id}^2\right) \circ X$$

- When composing linear-quadratic functions, max order is 4
- To get again linear-quadratic, ignore orders 3 and 4

Better jump integral

- Main idea: σ–localize the absolutely convergent integral $\xi * \mu^X$
  - $X \in \mathscr{V}^{\mathsf{d}} \iff X - X_0 = \mathrm{id} * \mu^X$
  - $X \in \mathscr{V}^{\mathsf{d}}_\sigma \iff X - X_0 = \mathrm{id} \star \mu^X$, see Č. and Ruf (2021b)
  - $\mathrm{id} \star \nu^X$ already defined in Kallsen (2004)
- The $\star \mu^X$ integral is closed under stochastic integration
- Every semimartingale X has a unique decomposition

$$X = X_0 + X^{\mathsf{qc}} + X^{\mathsf{dp}},$$

  - $X_0^{\mathsf{qc}} = X_0^{\mathsf{dp}} = 0$
  - X^{qc} is a quasi-left-continuous semimartingale
  - $X^{\mathsf{dp}} \in \mathscr{V}^{\mathsf{d}}_\sigma$ jumps only at predictable times

  We then have $[X^{\mathsf{qc}}, X^{\mathsf{dp}}] = 0$.
- These ideas are incorporated in $\mathfrak{I}(X)$, see Č. and Ruf (2022)
- Importantly, $\xi \circ X$ special iff both $\xi \circ X^{\mathsf{qc}}$ and $\xi \circ X^{\mathsf{dp}}$ special
- Simplifies drift calculations (and is algorithmic again)

Final quotes

Outline
Émery formula
Key properties
First example
Useful identities
Drift calculation
Lévy examples
Change of measure
Applications of mult. comp.
Pedagogical opportunities
Better jump integral
Final quotes
References

"Thus the parts of probability theory most relevant to [the question addressed here] are those results, usually abstract in appearance and French in origin, which are invariant under substitution of an equivalent measure."

— Harrison and Pliska (1981)

"Because in mathematics we pile inferences upon inferences, it is a good thing whenever we can subsume as many of them as possible under one symbol. For once we have understood the true significance of an operation, just the sensible apprehension of its symbol will suffice to obviate the whole reasoning process that earlier we had to engage anew each time the operation was encountered."

— Carl Jacobi (1804–1851)
source Kneser (1907) *transl.* Remmert (1991)

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