10.6 (a) The Itô formula dictates

$$
\begin{aligned}
\mathrm{d}\ln S_1(t) &= \left(0.1 - \frac{1}{2}0.04\right)\mathrm{d}t + 0.2\mathrm{d}B_1(t) \\
&= 0.08\mathrm{d}t + 0.2\mathrm{d}B_1(t) \\
\mathrm{d}\ln S_2(t) &= \left(0.15 - \frac{1}{2}0.16\right)\mathrm{d}t + 0.4\mathrm{d}B_2(t) \\
&= 0.07\mathrm{d}t + 0.4\mathrm{d}B_2(t)
\end{aligned}
$$

(b) Writing the above in integral form we obtain

$$
\begin{aligned}
\ln S_1(8) &= \ln S_1(0) + 0.08 \times 8 + 0.2\left(B_1(8) - B_1(0)\right) \\
\ln S_2(10) &= \ln S_2(0) + 0.07 \times 10 + 0.4\left(B_2(10) - B_2(0)\right)
\end{aligned}
$$

(c) Thus

$$
\begin{aligned}
\mathrm{E}_0\left[\ln S_1(8)\right] &= 0.64 \\
\mathrm{E}_0\left[\ln S_2(10)\right] &= 0.7 \\
\mathrm{Var}_0\left(\ln S_1(8)\right) &= 0.04 \times 8 = 0.32 \\
\mathrm{Var}_0\left(\ln S_2(10)\right) &= 0.16 \times 10 = 1.6
\end{aligned}
$$

(d) Since $\ln S_1(8)$ and $\ln S_2(10)$ are jointly normal, their difference will be normal. The mean and variance of the resulting distribution are

$$
\begin{aligned}
\mathrm{E}_0\left[\ln S_1(8) - \ln S_2(10)\right] &= 0.64 - 0.7 = -0.06 \\
\mathrm{Var}_0\left(\ln S_1(8) - \ln S_2(10)\right) &= \mathrm{Var}_0\left(\ln S_1(8)\right) + \mathrm{Var}_0\left(\ln S_2(10)\right) \\
&\quad -2\mathrm{Cov}_0\left(\ln S_1(8), \ln S_2(10)\right) \qquad (1)
\end{aligned}
$$

The covariance will take into account the contemporaneous correlation between the Brownian shocks,

$$
\begin{aligned}
\mathrm{Cov}_0(\ln S_1(8), \ln S_2(10)) &= 0.2 \times 0.4 \times \mathrm{Cov}_0\left(B_1(8) - B_1(0), B_2(10) - B_2(0)\right) \\
&= 0.08\mathrm{Cov}_0\left(B_1(8) - B_1(0), B_2(8) - B_2(0)\right) \\
&\quad +0.08\mathrm{Cov}_0\left(B_1(8) - B_1(0), B_2(10) - B_2(8)\right) \\
&= 0.08 \times 0.6 \times \\
&\quad \sqrt{\mathrm{Var}_0\left(B_1(8) - B_1(0)\right)}\sqrt{\mathrm{Var}_0\left(B_2(8) - B_2(0)\right)} \\
&= 0.08 \times 0.6 \times 8 = 0.384 \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) we have

$$
\begin{aligned}
\ln S_1(8) - \ln S_2(10)|\mathcal{F}_0 &\sim N(-0.06, 0.32 + 1.6 - 2 \times 0.384) \\
&\sim N(-0.06, 1.152).
\end{aligned}
$$

(e) We want to calculate

$$
P\left(\frac{S_1(8)}{S_2(10)} \geq 2\right)
$$

and this is equivalent to

$$\begin{aligned} P\left(\ln \frac{S_1(8)}{S_2(10)} \geq \ln 2\right) &= P(\ln S_1(8) - \ln S_2(10) \geq \ln 2) \\ &= P\left(\frac{\ln S_1(8) - \ln S_2(10) + 0.06}{\sqrt{1.152}} \geq \frac{\ln 2 + 0.06}{\sqrt{1.152}}\right) \\ &= 1 - \Phi\left(\frac{\ln 2 + 0.06}{\sqrt{1.152}}\right) = 0.241. \end{aligned}$$

(f) Similarly, we wish to find $R_{1\%}$ such that

$$0.01 = P\left(S_1(8)/S_2(10) \leq R_{1\%}\right).$$

We proceed by taking logarithms on both sides of the inequality and normalizing to obtain a standard normal variable on the left-hand side:

$$\begin{aligned} 0.01 &= P\left(\ln\left(S_1(8)/S_2(10)\right) \leq \ln R_{1\%}\right) \\ &= P\left(\frac{\ln\left(S_1(8)/S_2(10)\right) + 0.06}{\sqrt{1.152}} \leq \frac{\ln R_{1\%} + 0.06}{\sqrt{1.152}}\right). \end{aligned}$$

It becomes clear that $\left(\ln R_{1\%} + 0.06\right)/\sqrt{1.152}$ equals the 1% quantile of a standard normal distribution:

$$\begin{aligned} \frac{\ln R_{1\%} + 0.06}{\sqrt{1.152}} &= -2.326 \\ \ln R_{1\%} &= -2.326 \times \sqrt{1.152} - 0.06 = -2.557 \\ R_{1\%} &= \mathrm{e}^{-2.557} = 0.078. \end{aligned}$$