B.17 (a) By virtue of the portfolio theorem, for uncorrelated variables the variance of a sum equals sum of variances:

$$\begin{aligned}\mathrm{Var}(\triangle X_1 + \ldots + \triangle X_N) &= \mathrm{Var}(\triangle X_1) + \ldots + \mathrm{Var}(\triangle X_N) \\ &= N\sigma^2 \triangle t = \sigma^2.\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\mathrm{Var}(t_1\triangle X_1 + \ldots + t_N\triangle X_N) &= \mathrm{Var}(t_1\triangle X_1) + \ldots + \mathrm{Var}(t_N\triangle X_N) \\ &= t_1^2\mathrm{Var}(\triangle X_1) + \ldots + t_N^2\mathrm{Var}(\triangle X_N) \\ &= \left(t_1^2\triangle t + \ldots + t_N^2\triangle t\right)\sigma^2 \qquad (1)\end{aligned}$$

Substitute $t_k = k/N$ and $\triangle t = 1/N$ into (1) and use the formula $\sum_{k=1}^N k^2 = N(N+1)(2N+1)/6$:

$$\begin{aligned}\mathrm{Var}(t_1\triangle X_1 + \ldots + t_N\triangle X_N) &= \frac{\sigma^2}{N}\sum_{k=1}^N \frac{k^2}{N^2} \\ &= \sigma^2 \frac{N(N+1)(2N+1)}{6N^3}. \qquad (2)\end{aligned}$$

(c) $\mathrm{Var}(\triangle X_1 + \ldots + \triangle X_N) = \sigma^2$ no matter what N we take. The shocks $\triangle X_i$ were constructed to have this property.

(d) Using the result (2) we have

$$\begin{aligned}\lim_{N\to\infty} \mathrm{Var}(t_1\triangle X_1 + \ldots + t_N\triangle X_N) &= \lim_{N\to\infty} \sigma^2 N(N+1)(2N+1)/(6N^3) \\ &= \sigma^2 \lim_{N\to\infty} \frac{(1+\frac{1}{N})(2+\frac{1}{N})}{6} = \frac{\sigma^2}{3}. \qquad (3)\end{aligned}$$

(e) Recall from (1) that

$$\mathrm{Var}(t_1\triangle X_1 + \ldots + t_N\triangle X_N) = \left(t_1^2\triangle t + \ldots + t_N^2\triangle t\right)\sigma^2.$$

The expression $t_1^2\triangle t + \ldots + t_N^2\triangle t$ approximates the area under the curve t^2 between points 0 and 1, see Figure 1, therefore in the limit we obtain an integral

$$\lim_{N\to\infty} \mathrm{Var}(t_1\triangle X_1 + \ldots + t_N\triangle X_N) = \sigma^2 \int_0^1 t^2 \mathrm{d}t.$$

The evaluation of this integral just confirms the result previously obtained in (3):

$$\sigma^2 \int_0^1 t^2 \mathrm{d}t = \sigma^2 \left[\frac{t^3}{3}\right]_0^1 = \frac{\sigma^2}{3}.$$

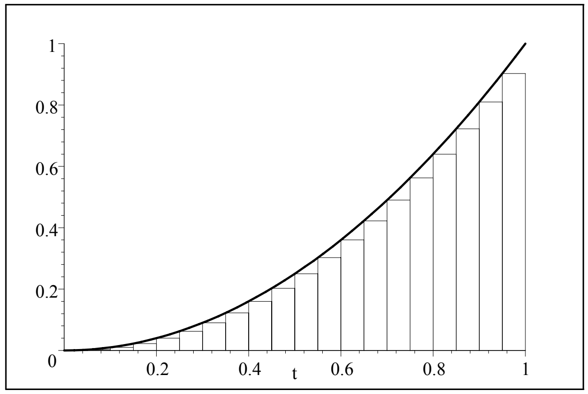

Figure 1: Area under the curve t^2 for $t \in [0,1]$, and its approximation by rectangular boxes representing the expression $t_1^2 \triangle t + \ldots + t_N^2 \triangle t$ for $N = 20$.

(f) By the portfolio theorem

$$\begin{aligned}
\mathrm{Cov}\left(\sum_{i=1}^{N} a(t_i)\triangle X_i, \sum_{j=1}^{N} b(t_j)\triangle X_j\right) &= \sum_{i=1}^{N}\sum_{j=1}^{N} a(t_i)b(t_j)\mathrm{Cov}\left(\triangle X_i, \triangle X_j\right) \\
&= \sum_{i=1}^{N} a(t_i)b(t_i)\mathrm{Var}\left(\triangle X_i\right) \\
&= \sigma^2 \sum_{i=1}^{N} a(t_i)b(t_i)\triangle t.
\end{aligned}$$

In the limit therefore

$$\lim_{N\to\infty} \mathrm{Cov}\left(\sum_{i=1}^{N} a(t_i)\triangle X_i, \sum_{j=1}^{N} b(t_j)\triangle X_j\right) = \sigma^2 \int_0^1 a(t)b(t)\mathrm{d}t.$$