B.26 The probability of (X_1, X_2) lying in a small square with bottom left corner (x_1, x_2) and dimensions $\triangle x_1 \times \triangle x_2$ is

$$P(\underbrace{x_1 < X_1 \leq x_1 + \triangle x_1, x_2 < X_2 \leq x_2 + \triangle x_2}_{\text{event } A})$$
$$= F(x_1 + \triangle x_1, x_2 + \triangle x_2) - F(x_1 + \triangle x_1, x_2) - F(x_1, x_2 + \triangle x_2) + F(x_1, x_2) \tag{1}$$

It is convenient to introduce vectors

$$e_1 = \begin{bmatrix} \triangle x_1 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ \triangle x_2 \end{bmatrix},$$

and the gradient and Hessian at the point (x_1, x_2):

$$\begin{aligned} \nabla^* &= \left[\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right], \\ H &= \begin{bmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_2} \end{bmatrix}. \end{aligned}$$

We can then approximate equation (1) using Taylor expansion to second order

$$\begin{aligned} P(A) &= F(x_1, x_2) + (e_1 + e_2)^* \nabla + \frac{1}{2}(e_1 + e_2)^* H (e_1 + e_2) \\ &\quad - \left(F(x_1, x_2) + e_1{}^* \nabla + \frac{1}{2} e_1{}^* H e_1 \right) \\ &\quad - \left(F(x_1, x_2) + e_2{}^* \nabla + \frac{1}{2} e_2{}^* H e_2 \right) \\ &\quad + F(x_1, x_2) + o((\triangle x_1)^2 + (\triangle x_2)^2) \end{aligned}$$

After cancelling the constant and linear terms we are left with the quadratic term:

$$\begin{aligned} P(A) &= \frac{1}{2}(e_1 + e_2)^* H (e_1 + e_2) - \frac{1}{2} e_1{}^* H e_1 - \frac{1}{2} e_2{}^* H e_2 \\ &\quad + o((\triangle x_1)^2 + (\triangle x_2)^2) \\ &= \frac{1}{2}\left(e_2{}^* H e_1 + e_1{}^* H e_2\right) + o((\triangle x_1)^2 + (\triangle x_2)^2) \\ &= \frac{\partial^2 F}{\partial x_1 \partial x_2} \triangle x_1 \triangle x_2 + o((\triangle x_1)^2 + (\triangle x_2)^2). \end{aligned}$$