Simplified stochastic calculus with examples for students

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Outline

- Simplified calculus from several perspectives
  - As a recipe for obtaining new formulae
  - As a useful shorthand
  - As a tool for practical calculations
  - As an algorithmic device
  - As a pedagogical tool
- Will touch upon

Émery, M. (1978). Stabilité des solutions des équations différentielles stochastiques application aux intégrales multiplicatives stochastiques. *Probab. Theory Related Fields* 41(3), 241–262.

Carr, P. and R. Lee (2013). Variation and share-weighted variation swaps on time-changed Lévy processes. *Finance Stoch.* *17*(4), 685–716.

Outline II

Based on a series of joint works with Johannes Ruf (LSE Math.)

- Intro for readers familiar with $\mathrm{d}t, \mathrm{d}W, \mathrm{d}N$ calculus
  - Simplified stochastic calculus with applications in Economics and Finance, *European J. Oper. Res. 293*(2), 2021, 547–560, `ssrn:3500384`.
  - Appendix on affine Riccati equations à la Duffie, Pan, and Singleton, `ssrn:3752072`.
- Theory at the level of Jacod and Shiryaev
  - Pure-jump semimartingales. *Bernoulli 27*(4), 2021, 2624–2648, `arXiv:1909.03020`.
  - Simplified stochastic calculus via semimartingale representations. *Electron. J. Probab. 27*, 2022, paper no. 3, `ssrn:3633638`.
  - Simplified calculus for semimartingales: Multiplicative compensators and changes of measure. To appear in *Stochastic Process. Appl.*, `ssrn:3633622`.
- Slides available from www.martingales.sk

Plan of the talk

- The Émery formula and drift calculation
- Samuelson's insight into geometric Brownian motion
- Generalized Yor formula (aka returns on NASDAQ$^\eta$)
- A plethora of applications
- The calculus in a nutshell

- NOTATION REMARK:
- We will encounter specific functions, such as
  - $x \mapsto x^2$
  - $x \mapsto e^x - 1$
  - $x \mapsto \log(1 + x)$, etc.
- The corresponding "function handles" will read
  - id^2
  - $e^{\mathrm{id}} - 1$
  - $\log(1 + \mathrm{id})$, etc.

Émery formula

- Semimartingale X ($\mathbb{R}$-valued for now)
- Function $\xi : \mathbb{R} \to \mathbb{R}$ in $\mathcal{C}^2$ with $\xi(0) = 0$
- Want to formalize a process with increment $\xi(\mathrm{d}X_t)$
- Suppose X has jumps of finite variation
- Continuous part $\mathrm{d}X^c := \mathrm{d}X - \Delta X$

$$\xi(\mathrm{d}X_t) = \xi'(0)\mathrm{d}X_t^c + \frac{1}{2}\xi''(0)\mathrm{d}[X^c, X^c]_t + \xi(\Delta X_t)$$

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Examples coming soon (in three slides)

Émery formula II

- Émery (1978) showed

$$\sum_{n\in\mathbb{N}} \xi\left(X^{t_n} - X^{t_{n-1}}\right) \xrightarrow{\text{u.c.p.}} \xi \circ X$$

  as the time partition $(t_n)_{n\in\mathbb{N}}$ becomes finer
- Hence $\xi \circ X = \int_0^{\cdot} \xi(\mathrm{d}X_t)$ is a ξ–variation of X!
- This result and Émery's $\xi(\mathrm{d}X_t)$ notation got lost somehow
- "G–variation" in Carr & Lee (2013) is the same concept but it cites Jacod (2008), who only proves convergence in Skorokhod topology

Definition 2.2 (G-variation) For $G \in \mathbb{V}(Y)$, define the G-variation of Y to be

$$\overbrace{V_t^{Y,G}}^{G\circ Y} := \cancel{\alpha_G \mathrm{TV}(Y^{\mathrm{d}})_t} + \overbrace{\beta_G}^{G'(0)}(Y_t - Y_0) + \overbrace{\gamma_G}^{1/2\, G''(0)}[Y^{\mathrm{c}}]_t + \sum_{0<s\le t}\left(G(\Delta Y_s) - \overbrace{\beta_G}^{G'(0)} \Delta Y_s\right), \quad (2.8)$$

Drift calculation I

- Need an easy, clean way to compute the drift of $\xi \circ X$
- Émery formula already provides this!

$$\xi \circ X = \xi'(0) \cdot X^c \ \ + \frac{1}{2}\xi''(0) \cdot [X,X]^c + \textstyle\sum_{t \leq \cdot} \ \xi(\Delta X_t)$$

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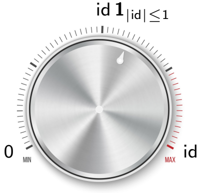
- Émery formula represents a spectrum of equivalent expressions



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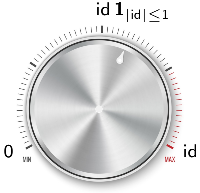

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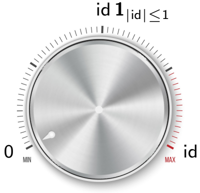

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  - $h = 0$ when X has finite variation jumps; $X[0] = X^c$

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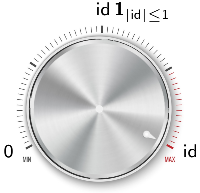

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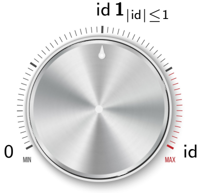

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  - $h = \mathrm{id}$ when X has finite drift; $X[\mathrm{id}] = X$
  - Otherwise $h = \mathrm{id}\,\mathbf{1}_{|\mathrm{id}| \leq 1}$ chops jumps at 1

Drift calculation II

- Suppose X is Lévy with Brownian vol σ and Lévy measure Π
- Drift rate also given: either $b^{X[0]}$, b^X, or $b^{X[1]}$
- Select h accordingly
- Assume $\xi \circ X$ has finite drift (is a special semimartigale). Then

$$\xi \circ X = \xi'(0) \cdot X[h] + \frac{1}{2}\xi''(0) \cdot [X,X]^c + \textstyle\sum_{t\leq \cdot}(\xi(\Delta X_t) - \xi'(0)h(\Delta X_t))$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$b^{\xi\circ X} = \xi'(0)\; b^{X[h]} + \frac{1}{2}\xi''(0) \times \sigma^2 \quad + \textstyle\int_{\mathbb{R}}(\xi(x) - \xi'(0)h(x))\Pi(\mathrm{d}x)$$

- EXAMPLE: $\xi = \mathrm{id}^2$, predictable quadratic variation rate
- EXAMPLE: $\xi = \mathrm{e}^{\mathrm{id}} - 1$, expected growth rate of stock price

$$b^{\mathrm{id}^2\circ X} = \sigma^2 + \int_{\mathbb{R}} x^2\Pi(\mathrm{d}x)$$

$$b^{(\mathrm{e}^{\mathrm{id}}-1)\circ X} = b^{X[h]} + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}}(\mathrm{e}^x - 1 - h(x))\Pi(\mathrm{d}x)$$

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Lévy process: classical vs new approach

- Classical notation

$$X_t = X_0 + \int_0^t \alpha \mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s + \int_0^t \int_{|x|\leq 1} x\widehat{N}(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{|x|>1} xN(\mathrm{d}s, \mathrm{d}x)$$

  - N is a Poisson jump measure
  - Π the corresponding Lévy measure
  - $\widehat{N}(\mathrm{d}t, \mathrm{d}x) = N(\mathrm{d}t, \mathrm{d}x) - \Pi(\mathrm{d}x)\mathrm{d}t$
  - $\alpha, \sigma \in \mathbb{R}$
- Simplified calculus: just record the triplet

$$\left(b^{X[1]} = \alpha, c^X = \sigma^2, F^X = \Pi\right).$$

- The top equation translates to the trivial statement

$$X_t = X_0 + \mathrm{id} \circ X_t$$

  - not specific to Lévy processes
  - completely measure-invariant

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Drift vs moments

- Cumulative drift (a.k.a. additive compensator)

$$\xi \circ X - B^{\xi \circ X} \in \mathscr{M}_{\mathrm{loc}}$$

- For Lévy processes we have

$$B_t^{\xi \circ X} = b^{\xi \circ X} t$$

- More generally, if $\xi \circ X$ is PII, then for all $t \geq 0$

$$\mathrm{E}[\xi \circ X_t] = B_t^{\xi \circ X}$$

- Higher moments: if PII X has zero drift, then

$$\mathrm{E}[(X_t - X_0)^2] = B_t^{\mathrm{id}^2 \circ X}$$

$$\mathrm{E}[(X_t - X_0)^3] = B_t^{\mathrm{id}^3 \circ X}$$

Summary of the key points so far

- The ξ-variation of X is given by the Émery formula
- Easy to remember as 2nd order Taylor + jumps $\xi(\Delta X)$
- Émery formula is measure-invariant — no need for predictable characteristics of X
- It is also universal — can be applied to any semimartingale X
- Cumulative drift $B^{\xi\circ X}$ is easily obtained from the Émery formula
- ξ can take many forms: powers, exponentials, logarithms, etc.
- $B^{\xi\circ X}$ is immediately useful when computing moments of PII X

NEXT STEPS

- We shall see how to redeploy the drift multiplicatively
- This leads to the main applications
- Also explains genesis of some useful ξ functions

Samuelson's insight into geometric BM - I

- Reading between the lines in Samuelson (1965),

$$\frac{\mathrm{d}S_t}{S_t} = \underbrace{\mu \mathrm{d}t + \sigma \mathrm{d}W_t}_{\mathrm{d}X_t} \;\Rightarrow\; \mathrm{E}[S_t] \text{ grows at rate } \mu \text{ regardless of } \sigma!$$

- For a special semimartingale X, we have $\mathrm{d}X_t = \mathrm{d}B^X_t + \mathrm{d}M^X_t$

Theorem (Č. & Ruf, 2023)

Let X be a special $\mathbb{C}$–valued semimartingale with independent increments. Then

$$\mathrm{E}[\mathscr{E}(X)_t] = \mathscr{E}(B^X)_t \qquad \textit{regardless of } M^X\,!$$

- $\mathscr{E}(X)$ is the value of an asset / fund
- X is the arithmetic rate of return of this asset / fund
- Expected growth rate = growth rate of expected value

Samuelson's insight into geometric BM - II

Previous results in this direction:

- Kallsen & Muhle-Karbe (2010, P 3.12), $\mathscr{E}(X) > 0$
- Cont & Tankov (2004, P 8.23), X Lévy

Without independent increments:

- Lépingle & Mémin (1978); assume additionally $\Delta B^X \neq -1$. Then

$$\mathscr{E}(X)/\mathscr{E}(B^X) \quad \text{is a local martingale}$$

Corollary ("Samuelson + Émery")

Let X be a $\mathbb{C}$–valued semimartingale with independent increments such that $\xi \circ X$ is special. Then

$$\mathrm{E}[\mathscr{E}(\xi \circ X)_t] = \mathscr{E}(B^{\xi \circ X})_t$$

Yor formula and its generalizations

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- Similarly: if X increases by 0.1, then the percentage increase in e^X reads $\mathrm{e}^{0.1} - 1 \approx 10.5\%$, hence

$$\mathrm{e}^{\eta(X - X_0)} = \mathscr{E}((\mathrm{e}^{\eta\,\mathrm{id}} - 1) \circ X)$$

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$$\mathscr{E}(X)^2 = \mathscr{E}(2X + [X, X])$$

- We have learned much more in fact:

$$\mathscr{E}(X)^\eta = \mathscr{E}(((1 + \mathrm{id})^\eta - 1) \circ X)$$

- Similarly: if X increases by 0.1, then the percentage increase in e^X reads $\mathrm{e}^{0.1} - 1 \approx 10.5\%$, hence

$$\mathrm{e}^{\eta(X - X_0)} = \mathscr{E}((\mathrm{e}^{\eta\,\mathrm{id}} - 1) \circ X)$$

- $\mathrm{E}[\mathscr{E}((\mathrm{e}^{iu\,\mathrm{id}} - 1) \circ X)_t]$ for $u \in \mathbb{R}$ yields Lévy–Khintchin!

Samuelson + Émery + Yor = lots of applications

- Classical financial setting
- $S > 0$ is the stock price
- Assume $\ln S =: X$ is a given Lévy process (but could be any PII)
- Characteristic triplet

$$\left(b^X[1] = \alpha, c^X = \sigma^2, F^X = \Pi\right)$$

- Specific values

$$\alpha = 0.1; \qquad \sigma = 0.15; \qquad \Pi = 150N\left(0, \frac{0.2^2}{150}\right)$$

- Moments

$$b^{\mathrm{id}\circ X} \approx 0.1; \qquad b^{\mathrm{id}^2\circ X} = 0.25^2; \qquad \frac{b^{\mathrm{id}^4\circ X}}{0.25^4} \approx 0.008$$

- Excess kurtosis is 0.008 years expressed in appropriate units

$$0.008 \text{ years} = 2 \text{ days} = 16 \text{ hours} = 960 \text{ minutes}$$

Characteristic function

- Take $v \in \mathbb{C}$
- We want to evaluate $\mathrm{E}[\mathrm{e}^{vX_t}]$
- Easily obtain the universal representation

$$\mathcal{L}(\mathrm{e}^{vX}) = (\mathrm{e}^{v\,\mathrm{id}} - 1) \circ X$$

- This yields

$$b^{\mathcal{L}(\mathrm{e}^{vX})} = \alpha v + \frac{1}{2}\sigma^2 v^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{vx} - 1 - vx\mathbf{1}_{|x|\leq 1}\right) \Pi(\mathrm{d}x),$$

- Exponential compensator (Lévy–Khintchin)

$$\mathrm{E}\left[\mathrm{e}^{v(X_t - X_0)}\right] = \mathrm{E}\left[\mathscr{E}((\mathrm{e}^{vx} - 1) \circ X)_t\right] = \exp\left(b^{\mathcal{L}(\mathrm{e}^{vX})} t\right)$$

- $b^{\mathcal{L}(\mathrm{e}^{vX})} t$ is the cumulant generating function of $X_t - X_0$

Maximization of exponential utility

- $X = \ln S$
- Cumulative yield on \$1 investment $R = \mathcal{L}(\mathrm{e}^X) = (\mathrm{e}^{\mathrm{id}} - 1) \circ X$
- Fixed dollar investment λ
- Utility of terminal wealth $-\mathrm{E}[\mathrm{e}^{-\lambda R_t}]$
- Need to find the drift of $\mathcal{L}(\mathrm{e}^{-\lambda R})$
- Use the composition rule to find

$$\mathcal{L}(\mathrm{e}^{-\lambda R}) = (\mathrm{e}^{-\lambda\,\mathrm{id}} - 1) \circ R = \left(\mathrm{e}^{-\lambda(\mathrm{e}^{\mathrm{id}}-1)} - 1\right) \circ X$$

- Evaluate the drift rate

$$b^{\mathcal{L}(\mathrm{e}^{-\lambda R})} = -\alpha\lambda + \frac{\sigma^2}{2}\left(\lambda^2 - \lambda\right) + \int_{\mathbb{R}} \left(\mathrm{e}^{-\lambda(\mathrm{e}^x-1)} - 1 + \lambda x \mathbf{1}_{|x|\leq 1}\right) \Pi(\mathrm{d}x).$$

- Expected utility is $-\exp\left(b^{\mathcal{L}(\mathrm{e}^{-\lambda R})} t\right)$

Minimal entropy martingale measure

- Denote optimal investment by λ_*
- Density of MEMM ${}^{\mathrm{d}\mathrm{Q}}/_{\mathrm{d}\mathrm{P}}$ is proportional to marginal utility $\mathrm{e}^{-\lambda_* R}$
- We want Q–drift of $\mathcal{L}(\mathrm{e}^{vX})$ to get the c.f. of X_t under Q
- By Girsanov the same as P–drift of

$$\mathcal{L}(\mathrm{e}^{vX}) + \left[\mathcal{L}(\mathrm{e}^{vX}), \mathcal{L}(\mathrm{e}^{-\lambda_* R})\right] \\ = (\mathrm{e}^{v\,\mathrm{id}} - 1) \circ X + (\mathrm{e}^{v\,\mathrm{id}} - 1)(\mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)} - 1) \circ X$$

- Evaluate the P–drift rate of $\xi \circ X$ with $\xi = (\mathrm{e}^{vx} - 1)\mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)}$

$$b_{\mathrm{Q}}^{\mathcal{L}(\mathrm{e}^{vX})} = b^{\xi \circ X} = \alpha v + \frac{\sigma^2}{2}\left(v^2 - 2\lambda_* v\right) \\ + \int_{\mathbb{R}} \left((\mathrm{e}^{vx} - 1)\mathrm{e}^{-\lambda_*(\mathrm{e}^x - 1)} - vx\mathbf{1}_{|x|\leq 1}\right)\Pi(\mathrm{d}x),$$

Change of measure in classical notation

- Lengthy and difficult
- Find an explicit expression for $\log Z$

$$\log Z = -\int_0^\cdot \lambda_* \sigma \mathrm{d}W_s - \frac{1}{2}\int_0^\cdot \lambda_*^2 \sigma^2 \mathrm{d}s + \int_0^\cdot \int_{\mathbb{R}} -\lambda_*(\mathrm{e}^x - 1)\widehat{N}(\mathrm{d}s, \mathrm{d}x)$$
$$+ \int_0^\cdot \int_{\mathbb{R}} \left(-\lambda_*(\mathrm{e}^x - 1) - \left(\mathrm{e}^{-\lambda_*(\mathrm{e}^x-1)} - 1\right)\right) \Pi(\mathrm{d}x)\mathrm{d}s$$

- Construct a new Brownian motion for the measure Q,

$$\mathrm{d}W_t^{\mathrm{Q}} = \mathrm{d}W_t + \lambda_* \sigma \mathrm{d}t$$

- New compensated Poisson jump measure

$$\widehat{N}^{\mathrm{Q}}(\mathrm{d}t, \mathrm{d}x) = \widehat{N}(\mathrm{d}t, \mathrm{d}x) + \left(1 - \mathrm{e}^{-\lambda_*(\mathrm{e}^x-1)}\right) \Pi(\mathrm{d}x)\mathrm{d}t$$

  - Applebaum (2009), Theorem 5.2.12, Exercise 5.2.14
  - Øksendal & Sulem (2007), Theorem 1.32 and Lemma 1.33
- Substitute for W, N, and $\hat{N}$ in the original expression for X
- More examples in this vein Č. & Ruf (2021c)

Further applications (briefly)

- Knowledge of drift has many applications outside PII setting: PIDEs; HJB equations; Feynman–Kac; affine Riccati equations
- In PII setting
  - Make up your own Lévy measure
  - Pricing of simple contracts $\log S$ or $[\mathcal{L}(S), \mathcal{L}(S)]$
  - But even plain vanilla options via Fourier transform
  - A variety of risk-neutral measures Esscher, MEMM, VOMM (MMM), q-optimal, e.g.,

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{\mathscr{E}\left(-a(e^{\mathrm{id}}-1)\circ X\right)_T}{\mathscr{E}\left(B^{-a(e^{\mathrm{id}}-1)\circ X}\right)_T}, \qquad \text{with } a = \frac{b^{(e^{\mathrm{id}}-1)\circ X}}{b^{(e^{\mathrm{id}}-1)^2\circ X}}$$

  - Correlation between $\mathscr{E}(\eta\mathcal{L}(S))$ and S^η (equals 1 in B–S model)

$$\frac{\exp\left(b^{\eta(e^{\mathrm{id}}-1)(e^{\eta\,\mathrm{id}}-1)\circ X}\right)-1}{\sqrt{\exp\left(b^{\eta^2(e^{\mathrm{id}}-1)^2\circ X}\right)-1}\sqrt{\exp\left(b^{(e^{\eta\,\mathrm{id}}-1)^2\circ X}\right)-1}}$$

Applications of multiplicative compensators

- Proofs of moment bounds
  - to show existence and uniqueness of BSDE solutions, e.g., Kazi-Tani et al. (2015), Lemma A.5
  - to estimate variation distance of probability measures Kabanov et al. (1986), Theorem 2.1
  - to prove uniform integrability of local martingales, e.g., Lépingle & Mémin (1978), Lemma I.4; Ruf (2013), Corollary 5
- Filtration extension / shrinkage
  - e.g., Nikeghbali & Yor (2006), Section 4; Kardaras (2015); Aksamit & Jeanblanc (2017), Chapter 5; Kardaras & Ruf (2020), Section 5
- Theory of Markov processes
  - e.g., Itô & Watanabe (1965) Chapter 2; Chen et al. (2004), Theorem 3.1

Why are variations a relatively unused concept?

- For continuous X, all variations are linear–quadratic, e.g.,

$$(e^{\mathrm{id}} - 1) \circ X = (\mathrm{id} + \tfrac{1}{2}\,\mathrm{id}^2) \circ X$$

- Expression $\xi \circ X$ arises often but in contexts that have nothing to do with variations, e.g.,
  - Goll & Kallsen (2000, Lemma A.8); $S, S_- > 0$

$$\mathcal{L}(S) := \int_0^{\cdot} \frac{\mathrm{d}S_u}{S_{u-}} = (e^{\mathrm{id}} - 1) \circ \ln S$$

  - Mémin (1978, Proposition I-1); $\Delta Y \neq -1$

$$\frac{\mathscr{E}(X)}{\mathscr{E}(Y)} = \mathscr{E}\left(\left(\frac{1+\mathrm{id}_1}{1+\mathrm{id}_2} - 1\right) \circ (X, Y)\right)$$

  - Doléans-Dade (1970, Théorème 1) $\mathscr{E}(X) = e^{\ln(1+\mathrm{id})\circ X}$

Émery formula for complex functions

- Want $\mathbb{C}^n$–valued ξ applied to $\mathbb{C}^d$–valued X
- Define lifts $\hat{\mathrm{id}} : \mathbb{C}^m \to \mathbb{R}^{2m}$ and $\check{\mathrm{id}} : \mathbb{C}^m \to \mathbb{C}^{2m}$

$$\hat{\mathrm{id}} = (\mathrm{Re}\,\mathrm{id}_1, \mathrm{Im}\,\mathrm{id}_1, \ldots, \mathrm{Re}\,\mathrm{id}_m, \mathrm{Im}\,\mathrm{id}_m)$$
$$\check{\mathrm{id}} = (\mathrm{id}_1, \mathrm{id}_1^*, \ldots, \mathrm{id}_m, \mathrm{id}_m^*)$$

- Real derivatives $\hat{D}$; Wirtinger derivatives $\check{D}$

$$\begin{aligned}\xi \circ X &= \hat{D}\xi(0) \cdot \hat{X} + \frac{1}{2}\hat{D}^2\xi(0) \cdot [\hat{X}, \hat{X}]^c + (\xi - \hat{D}\xi(0)\hat{\mathrm{id}}) * \mu^X \\ &= \check{D}\xi(0) \cdot \check{X} + \frac{1}{2}\check{D}^2\xi(0) \cdot [\check{X}, \check{X}]^c + (\xi - \check{D}\xi(0)\check{\mathrm{id}}) * \mu^X\end{aligned}$$

- If ξ is analytic or ξ, X real-valued, we can drop ˆ and ˇ
- Definition of "$\circ$" handles restricted domains, e.g., $\log(1 + \mathrm{id}) \circ X$ makes sense if $\Delta X > -1$

Example of a useful non-analytic representation

- Consider $\xi = |1+\mathrm{id}\,|^\alpha - 1$ for $\alpha \in \mathbb{C}$
- On a sufficiently small neighbourhood of zero

$$|1+\mathrm{id}|^\alpha - 1 = (1+\mathrm{id})^{\frac{\alpha}{2}}(1+\mathrm{id}^*)^{\frac{\alpha}{2}} - 1.$$

- Apply formal Wirtinger calculus to obtain, e.g.,

$$\partial_x \xi = \frac{\alpha}{2}(1+\mathrm{id})^{\frac{\alpha}{2}-1}(1+\mathrm{id}^*)^{\frac{\alpha}{2}}; \;\; \partial_{x^*} \xi = \frac{\alpha}{2}(1+\mathrm{id})^{\frac{\alpha}{2}}(1+\mathrm{id}^*)^{\frac{\alpha}{2}-1};$$

- Émery formula ($\Delta X \neq -1$)

$$(|1+\mathrm{id}|^\alpha - 1) \circ X = \alpha \cdot \operatorname{Re} X + \frac{\alpha}{2}(\alpha-1)[\operatorname{Re} X, \operatorname{Re} X]^c + \frac{\alpha}{2}[\operatorname{Im} X, \operatorname{Im} X]^c$$
$$+ \sum_{t\leq \cdot} (|1+\Delta X_t|^\alpha - 1 - \alpha \operatorname{Re} \Delta X_t)$$

Mellin transform of signed stochastic exponential

- Cannot be tackled by existing tools
- For fixed $\alpha \in \mathbb{C}$ define

$$f_1 = |\mathrm{id}|^\alpha \mathbf{1}_{\mathrm{id}\neq 0}; \quad f_2 = |\mathrm{id}|^\alpha \left(\mathbf{1}_{\mathrm{id}>0} - \mathbf{1}_{\mathrm{id}<0}\right); \quad \xi_{1,2} = f_{1,2}(1+\mathrm{id})-1$$

- For all $\mathbb{R}$–valued Y,

$$f_{1,2}(\mathscr{E}(Y)) = \mathscr{E}(\xi_{1,2} \circ Y)$$

- Observe $f_1 + f_2 = (\mathrm{id}^+)^\alpha$ and $f_1 - f_2 = (\mathrm{id}^-)^\alpha$
- If Y is PII, we get Mellin transforms of $\mathscr{E}(Y)_t^+$ and $\mathscr{E}(Y)_t^-$

$$\mathrm{E}[f_{1,2}(\mathscr{E}(Y)_t)] = \mathscr{E}\left(B^{\xi_{1,2}\circ Y}\right)_t$$

- Lévy–Khintchin is of no use here
- Apply this calculation to exponential Lévy MV portfolio

MV wealth as a signed stochastic exponential I

- Merton model log return X with triplet

$$(b^{X[0]} = \mu, \sigma^2, \Pi = \lambda\Phi(0, \gamma^2))$$

- Parameter values $\mu = 0.2$, $\sigma = 0.2$, $\lambda = 1$, $\gamma = 0.1$, and zero interest rate
- Optimal wealth $1 - \mathscr{E}(\underbrace{-a(\mathrm{e}^{\mathrm{id}} - 1) \circ X}_{Y})$, where

$$a = \frac{b^{(\mathrm{e}^{\mathrm{id}}-1)\circ X}}{b^{(\mathrm{e}^{\mathrm{id}}-1)^2\circ X}} = \frac{\mu + \sigma^2/2 + \lambda(\mathrm{e}^{\gamma^2/2} - 1)}{\sigma^2 + \lambda(\mathrm{e}^{2\gamma^2} - 2\mathrm{e}^{\gamma^2/2} + 1)} \approx 4.48;$$

- Evaluate the exponential compensators

$$b^{\xi_1(\mathrm{id};\alpha)\circ Y} = b^{\xi_1(-a(\mathrm{e}^{\mathrm{id}}-1);\alpha)\circ X} = I_1(\alpha);$$
$$b^{\xi_2(\mathrm{id};\alpha)\circ Y} = b^{\xi_2(-a(\mathrm{e}^{\mathrm{id}}-1);\alpha)\circ X} = I_1(\alpha) - 2I_2(\alpha),$$

MV wealth as a signed stochastic exponential II

- Auxiliary expressions

$$
\begin{aligned}
l_1(\alpha) = & -\alpha a\left(\mu + \frac{1}{2}(1+a)\sigma^2\right) + \frac{1}{2}\alpha^2(a\sigma)^2 \\
& + \int_{\mathbb{R}} \left(|1 - a(e^x - 1)|^\alpha \mathbf{1}_{a(e^x-1)\neq 1} - 1\right)\Pi(dx); \\
l_2(\alpha) = & \int_{\mathbb{R}} |1 - a(e^x - 1)|^\alpha \mathbf{1}_{a(e^x-1)>1}\Pi(dx)
\end{aligned}
$$

- Evaluate the Mellin transforms

$$
g_+(\alpha) = \mathrm{E}\left[|\mathscr{E}(Y)_t|^\alpha \mathbf{1}_{\{\mathscr{E}(Y)_t>0\}}\right] = e^{l_1(\alpha)T}\frac{1 + e^{-2l_2(\alpha)T}}{2};
$$

$$
g_-(\alpha) = \mathrm{E}\left[|\mathscr{E}(Y)_t|^\alpha \mathbf{1}_{\{\mathscr{E}(Y)_t<0\}}\right] = e^{l_1(\alpha)T}\frac{1 - e^{-2l_2(\alpha)T}}{2}
$$

- Observe $g_-(0) = \mathrm{P}(\mathscr{E}(Y) < 0) \approx 2.2\%$

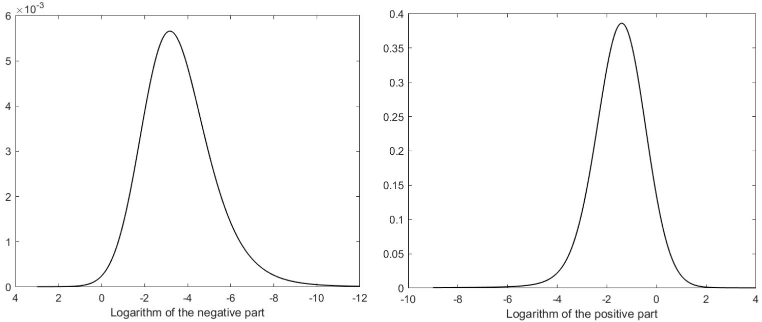
MV wealth as a signed stochastic exponential III

- Compute subdensities of $\log|\mathscr{E}(Y)|$ conditional on $\mathscr{E}(Y) \gtrless 0$ by Fourier inversion of conditional c.f.-s

$$\phi_+(u) = \frac{g_+(iu)}{g_+(0)}; \qquad \phi_-(u) = \frac{g_-(iu)}{g_-(0)}, \qquad u \in \mathbb{R},$$

- The whole computation is structured and algorithmic

MV wealth as a signed stochastic exponential IV



(a) Subdensity of $\log \mathscr{E}(-a(e^{\mathrm{id}} - 1) \circ X)^-_T$. (b) Subdensity of $\log \mathscr{E}(-a(e^{\mathrm{id}} - 1) \circ X)^+_T$.

Figure: Distribution of a signed stochastic exponential

MV wealth as a signed stochastic exponential V

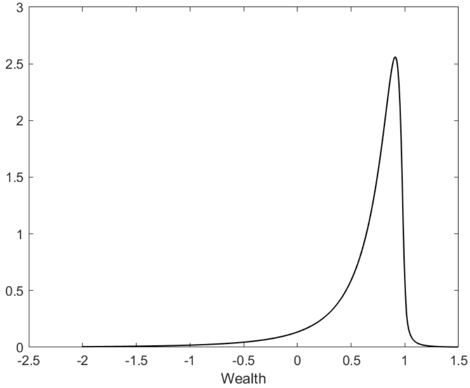


Figure: Density of the terminal wealth distribution $1 - \mathscr{E}(-a(e^{\mathrm{id}} - 1) \circ X)_T$.

Appeal

- We are collecting examples to include in a book
- Let us know of other applications
- Could be the same maths in different context (e.g., Act. Sci.)
- New applications (e.g., recursive utility)

The calculus in a nutshell

- Make ξ predictable: calculus of "predictable variations"
- More accurately: semimartingale representations
- Integral is a "linear variation"

$$\zeta \cdot X = (\zeta \,\mathrm{id}) \circ X$$

- The same Émery formula applies

$$\xi \circ X = \xi'(0) \cdot X + \frac{1}{2}\xi''(0) \cdot [X,X]^c + \sum_{t \leq \cdot} (\xi_t(\Delta X_t) - \xi_t'(0)\Delta X_t)$$

- Each of the three integrals must exist separately
  - $\xi'(0) \in L(X)$
  - $\xi''(0) \in L([X,X]^c)$
  - $\xi(\Delta X) - \xi'(0)\Delta X$ absolutely summable

Observe $\xi^f := f(X_- + \mathrm{id}) - f(X_-)$ yields the Itô–Meyer formula

$$\xi^f \circ X = f'(X_-) \cdot X + \frac{1}{2} f''(X_-) \cdot [X,X]^c + \sum_{t \leq \cdot} (f(X_t) - f(X_{t-}) - f'(X_{t-})\Delta X_t)$$

Universal representations

- Want the calculus to be rigorous, flexible, and easy to use
- Need a rich class of ξ, where nothing strange can happen

Definition (Universal representing functions)

$\mathfrak{U}$ denotes the set of predictable functions ξ such that, P–a.s.,

(i) $\xi_t(0) = 0$, for all $t \geq 0$.

(ii) $x \mapsto \xi_t(x)$ is twice real-differentiable at zero, for all $t \geq 0$.

(iii) $D\xi(0)$ and $D^2\xi(0)$ are locally bounded.

(iv) There is a predictable locally bounded process $K > 0$ such that

$$\sup_{0<|x|\leq 1/K} \frac{|\xi(x) - D\xi(0)x|}{|x|^2} \text{ is locally bounded.}$$

Key properties

- $\mathfrak{U}$ is closed under common operations
- Starting from $X = X_0 + \mathrm{id} \circ X$ and using only
  - composition, i.e., the "$\circ$" operation with functions in $\mathfrak{U}$;

$$\psi \circ (\xi \circ X) = \psi(\xi) \circ X$$

  - change of variables by means of (deterministic) $\mathcal{C}^2$ functions

$$f(X) = f(X_0) + (f(X_- + \mathrm{id}) - f(X_-)) \circ X;$$

  - locally bounded integration;

$$\zeta \cdot (\xi \circ X) = \zeta\xi \circ X$$

  every result will be of the form $\eta \circ X$ for some $\eta \in \mathfrak{U}$
- In practice, we never leave $\mathfrak{U}$, so no checking necessary

Example: how to derive a new representation

- Suppose X is log return
- Cumulative rate of return is then $\mathcal{L}(e^X) := e^{-X_-} \cdot e^X$
- Let us compute $[\mathcal{L}(e^X), \mathcal{L}(e^X)]$
- Proceed in steps
  - Change of variables

$$e^X = e^{X_0} + (e^{X_- + \mathrm{id}} - e^{X_-}) \circ X$$

  - Locally bounded integration

$$\mathcal{L}(e^X) = e^{-X_-} \cdot e^X = e^{-X_-}(e^{X_- + \mathrm{id}} - e^{X_-}) \circ X = (e^{\mathrm{id}} - 1) \circ X$$

  - Composition

$$[\mathcal{L}(e^X), \mathcal{L}(e^X)] = \mathrm{id}^2 \circ \mathcal{L}(e^X) = (e^{\mathrm{id}} - 1)^2 \circ X$$

- Instead of manipulating complicated **stochastic** expressions ✘ one performs simple **algebraic** operations ✓

Beyond universal representations

- One can move beyond universal representations
- $\mathfrak{I}(X)$ are functions specific to X such that $\xi \circ X$ makes sense
  - E.g., integrand ζ unbounded: $\zeta\,\mathrm{id} \notin \mathfrak{U}$ but $\zeta\,\mathrm{id} \in \mathfrak{I}(X)$
- Improvements to the Émery formula: better jump integral $\star$, no differentiability at predictable jump times
- General composition theorem: Let $\xi \in \mathfrak{I}(X)$, $\psi \in \mathfrak{I}(\xi \circ X)$, and

$$\psi'(0) \in L(\xi''(0) \cdot [X,X]^c) \cap L\left((\xi - \xi'(0)\,\mathrm{id}) \star \mu^X\right)$$

Then $\psi(\xi) \in \mathfrak{I}(X)$ and we have

$$\psi \circ (\xi \circ X) = \psi(\xi) \circ X$$

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Then $\psi(\xi) \in \mathfrak{I}(X)$ and we have

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  - $\xi = \zeta\,\mathrm{id}$ ✓ generalizes associative property of SI

Better jump integral

Pure-jump semimartingales. *Bernoulli 27*(4), 2021, 2624–2648, `arXiv:1909.03020`.

- The Émery formula features one absolutely convergent sum in contrast to one non-absolutely convergent integral

There is a way to sum jumps at predictable times non-absolutely

- This corresponds to σ–localizing the absolutely convergent sum
- The new summation can sometimes be done at inaccessible times but it always works at predictable times
- The calculus at predictable times is super well-behaved
- New semimartingale decomposition

$$X = X_0 + X^{\mathrm{qc}} + X^{\mathrm{dp}},$$

  - X^{qc} is a quasi-left-continuous semimartingale
  - X^{dp} equals the sum of its jumps at predictable times

Furthermore, $[X^{\mathrm{qc}}, X^{\mathrm{dp}}] = 0$.

Consequences for representations

- Suppose $\mathcal{T}_X$ exhausts jumps of X^{dp} and let

$$\xi \circ X^{\mathrm{dp}} := \sum_{\tau \in \mathcal{T}_X} \xi_\tau(\Delta X_\tau)$$

- Define $\xi \circ X^{\mathrm{qc}}$ by the Émery formula (with $\star$ instead of $*$)
- Let $\xi \circ X = \xi \circ X^{\mathrm{qc}} + \xi \circ X^{\mathrm{dp}}$
- $\xi \circ X$ is special iff both $\xi \circ X^{\mathrm{qc}}$ and $\xi \circ X^{\mathrm{dp}}$ special
- Simplifies drift calculations

$$B^{\xi \circ X^{\mathrm{dp}}} = \sum_{\tau \in \mathcal{T}_X} \mathrm{E}_{\tau-}[\xi_\tau(\Delta X_\tau)]$$

Pedagogical opportunities, continuous X

- For continuous X it is common to write

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2$$

- In Émery's notation literally $(\mathrm{d}X_t)^2 = \mathrm{d}[X,X]_t$
- McKean (1969) suggested the heuristics $\mathrm{d}W_t\mathrm{d}t = 0$, $(\mathrm{d}t)^2 = 0$
- Better rule: $(\mathrm{d}X_t)^3 = 0$, $(\mathrm{d}X_t)^4 = 0$ for any continuous X
- Why useful: for continuous X

$$\xi \circ X = \left(\xi'(0)\,\mathrm{id} + \tfrac{1}{2}\xi''(0)\,\mathrm{id}^2\right) \circ X$$

- When composing linear-quadratic functions, max order is 4
- To get again linear-quadratic, ignore orders 3 and 4
- Also useful for small jumps asymptotics

Final quotes

"Thus the parts of probability theory most relevant to [the question addressed here] are those results, usually abstract in appearance and French in origin, which are invariant under substitution of an equivalent measure."

— Harrison & Pliska (1981)

"Because in mathematics we pile inferences upon inferences, it is a good thing whenever we can subsume as many of them as possible under one symbol."

— Carl Jacobi (1804–1851)

source Kneser (1907) *transl.* Remmert (1991)

"As often happens in the history of science, the simple ideas are the hardest to achieve; simplicity does not come of itself but must be created."

— Truesdell (1960)

comment on the work of Leonhard Euler

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